Section and problem #: Chapter 1
Triangular Numbers
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<u>Question</u>: What is the formula that generates the triangular numbers (1, 3, 6, 10, 15, ...)?

<u>Hypothesis</u>: The formula is:

$$\frac{m(m+1)}{2} \quad \text{for } m = 1, 2, 3, 4, \ldots$$

<u>Does the hypothesis fit the data</u>?

$$T_m = \frac{m(m+1)}{2}$$

$$T_1 = \frac{1(1+1)}{2} = 1 \checkmark$$

$$T_2 = \frac{2(2+1)}{2} = 3 \checkmark$$

$$T_3 = \frac{3(3+1)}{2} = 6 \checkmark$$

$$T_4 = \frac{4(4+1)}{2} = 10 \checkmark$$

$$T_5 = \frac{5(5+1)}{2} = 15 \checkmark$$

Yes, the hypothesis fits the data.

<u>Statement</u>: Prove that $T_m = \frac{m(m+1)}{2}$

<u>Observations</u>: We observed that the formula is related to the area of a triangle, and this can be proved geometrically. First, we found the area of the triangles with length of sides m, but we noticed that something is missing in each of them. These observations appear on the next page.

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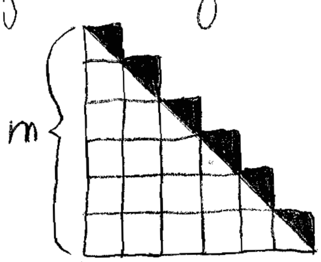
Observations: (continued)

		area $\left(\frac{m^2}{2}\right)$	missing factor
T_1	1	1/2	1/2
T_2	3	2	1
T_3	6	4.5	1.5
T_4	10	8	2
T_5	15	12.5	2.5
T_6	21	18	3
T_7	28	24.5	3.5
T_8	36	32	4
⋮			

What is area formula $\left(\frac{m^2}{2}\right)$ missing?

$$\frac{m(m+1)}{2} = \frac{m^2+m}{2} = \frac{m^2}{2} + \frac{m}{2} \longrightarrow \text{MISSING FACTOR!}$$

First Proof: In this approach we attempt to prove the formula by representing the solution for Tm as an arrangement of squares in a triangle. We treated the side of each square as a unit length in order to find the area (and hopefully the number of squares there are) using the area formula for a triangle ($\frac{1}{2}ab$). In our triangle, side $a=b=m$, so $\frac{1}{2}ab = \frac{m^2}{2}$.



When you do this (especially on grid paper), you notice that the shaded region is not counted. So, if you count the number of boxes and add them up, you'll find that the remaining area is $m/2$, because there are m boxes that were cut in half.

∴ The resulting formula for the number of boxes in this diagram is $\frac{1}{2}(m^2) + m/2$, which is equivalent to saying $\frac{m(m+1)}{2}$.

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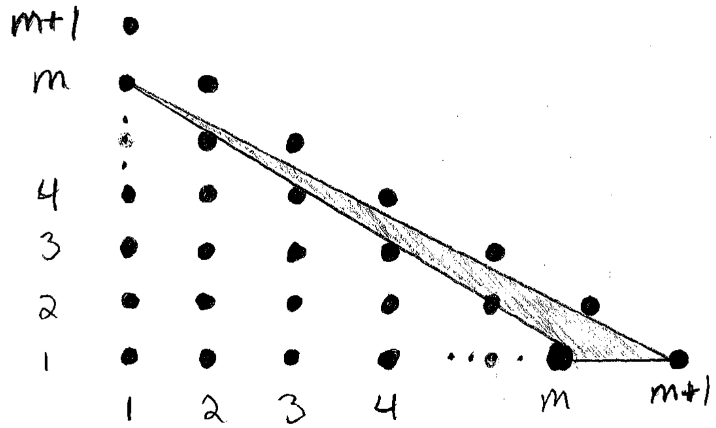
Geometric Proof:



We want to prove the formula is $\frac{m(m+1)}{2}$

First, we used $m^2/2$ to find the area of the triangle with with side lengths m. Then we notice that there is a piece missing (the shaded triangle)

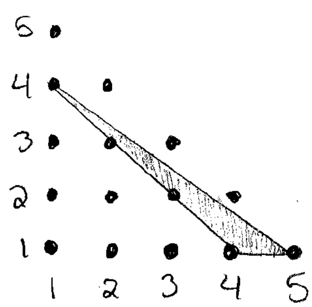
The area of the shaded triangle is:

$$\frac{[(m+1)-m]m}{2} = \boxed{\frac{m}{2}} \longrightarrow \text{THE MISSING AREA}$$

$\therefore$ The total area is $m^2/2 + m/2 = \frac{m(m+1)}{2} = T_m$

Example:

$T_4 = 10$



$$\left[\frac{m(m+1)}{2} = \frac{m^2+m}{2} = \frac{m^2}{2} + \frac{m}{2}\right]$$

$\frac{m^2}{2} = \frac{4^2}{2} = 8$ This is the area of a triangle w/ sides = 4

$\frac{m}{2} = \frac{4}{2} = 2$ This is the area of the shaded region

$\frac{m^2}{2} + \frac{m}{2} = 10$ ✓