Triangular Numbers: Induction

The two formulas for the sequence: 1, 3, 6, 10, 15, 21, 28, 36 are:

1) Closed: \( T_n = \frac{m(m+1)}{2} \) \( m \) starts at 1

2) Recursive: \( T_1 = 1 \) \( T_n = T_{n-1} + n \) \( n \) starts at 2

Through the process of mathematical induction it is possible to prove these equations to be correct for all cases.

**Base Case:** let \( m = 2 \)

Using a small number in the equation \( \frac{m(m+1)}{2} \)

\[
T_2 = \frac{2(2+1)}{2} = \frac{2(3)}{2} = \frac{6}{2} = 3
\]

This proves that at spot 2, the value is 3

\( \Rightarrow \) So the formula

\[ T_m = \frac{m(m+1)}{2} \] holds when \( m = 2 \)

Now let's assume the formula holds up to \( n-1 \), i.e.

We substitute \( (n-1) \) from the recursive formula for \( m \) in the closed formula and simplify.

\[
T_{n-1} = \frac{(n-1)(n-1+1)}{2} = \frac{n^2 - n}{2}
\]
So: \( T_n = T_{n-1} + n \)  

\[
\frac{n(n+1)}{2} = \frac{n^2 - n}{2} + n
\]  

\[
\frac{n^2 + n}{2} = \frac{n^2 - n}{2} + \frac{2n}{2}
\]  

\[
\frac{n^2 + n}{2} = \frac{n^2 - n}{2} + \frac{2n}{2}
\]  

\[
\frac{n^2 + n}{2} = \frac{n^2 + n}{2} \checkmark
\]

\[
T_n = \frac{n(n+1)}{2}
\]
Extension

is \( m = n - 2 \)

\[
\frac{(n-2)[(n-2)+1]}{2}
\]

\[
\frac{n^2 - 3n + 2}{2}
\]

\( n = 3 \rightarrow \frac{3^2 - 3(3) + 2}{2} = \frac{1}{2} \)  

\( n = 4 \rightarrow \frac{4^2 - 3(4) + 2}{2} = \frac{6}{2} = 3 \)  

still shows pattern of triangular #'s

If \( m = n - 3 \)

\[
\frac{(n-3)[(n-3)+1]}{2}
\]

\[
\frac{(n-3)(n-2)}{2}
\]

\[
\frac{n^2 - 5n + 6}{2}
\]

\( n = 4 \rightarrow \frac{16 - 20 + 6}{2} = \frac{2}{2} = 1 \)  

\( n = 5 \rightarrow \frac{25 - 25 + 6}{2} = \frac{6}{2} = 3 \)  

shows pattern of triangular #'s