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Final Draft

Section 1 Problem 1.2
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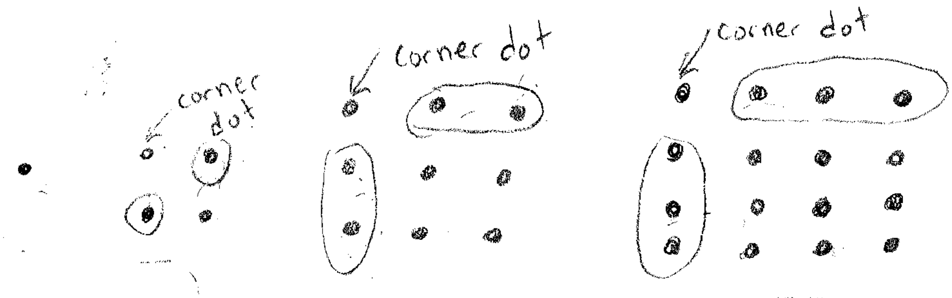
Question: Try adding up the first few odd numbers and see if the numbers you get satisfy some sort of pattern. Once you find the pattern, express it as a formula. Give a geometric verification that your formula is correct.

When the first few odd numbers were added together, a pattern was discovered. 1+3=4 and when 4, the sum of the first two odd numbers, is added to the next odd number, 5, 9 is the resulting sum. The next odd number after 5 is 7, so when that is added to 9, 16 is the sum, and when 9 is added to 16, the sum is 25. The pattern that can be seen here is that when these numbers are added together, their resulting sums are perfect squares. Therefore, a formula $f_n = n^2$ is derived. F_n is the number that one wants to get when they plug a number, n, into the equations, where $n_1 = 2$, $n_2 = 3$, $n_3 = 4 \ldots n_n = n+1$. For example, if you wanted to figure out what the 5th number in this sequence was, n_5, you would simply take 5+1 and put it into the equation n^2, which yields 36.

This formula works geometrically. Look at the following perfect squares represented by geometric dots:



To create the next square, you have to add an extra layer to the square, both to the columns and rows, with each side always increasing by 1, plus a corner. Because you have to add an extra layer to each side, you're adding 2, 4, 6, 8, 10…plus the corner. Therefore, you are simply adding consecutive odd numbers. For example, if one looks at the fourth square, it can be seen that there are a total of 4 dots in each the first column and the first row, but when the corner dot on the upper left hand corner of the square is eliminated, there is a total of 3 dots in the first row and column. When those two groups are added together, plus the corner dot, this gives us the odd number 7. When the same procedure is done on the square that is 3x3, this gives us 5, and the previous square yields 3. Therefore, it can be seen that through each geometrically perfect square, one is simply adding consecutive numbers by using the procedure that was just described.

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Extension

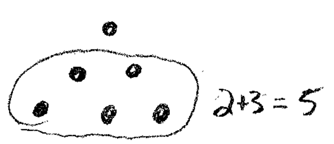
An extension similar to the original problem can be seen here. What happens if you add up every other odd number to the sum of the previous two numbers? So, to start out with, you would take the first number 1, skip and odd number and add it to the next odd number, which is 5. This yields 6 (1+5=6). Then taking that sum, you would add the next odd number that occurs after you skip an odd number. So, the next odd number after 5, which is 7, is skipped and 9 is added to 6. This gives us 15 (9+6=15). 15 is then added to 13 because 13 is the next odd number that occurs after skipping 11, which is the next immediate odd number after 9 (15+13=28). The next numbers would be 28+17=45 and 45+21=66. Here is a visual of what is happening:

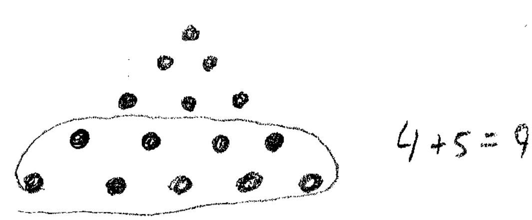
1+5=6→6 is the first triangular number that occurs
(10 is skipped)
6+9 = 15→15 is the third triangular number that occurs
(21 is skipped)
15+13=28→28 is the fifth triangular number that occurs
(36 is skipped)
28+17=45→45 is the seventh triangular number that occurs
(55 is skipped)
45+21=66→66 is the ninth triangular number that occurs

By looking at these five examples, the pattern can be seen more easily. In the first example, 1 and the 3rd odd number, 5, are added because when you add, you need to skip an odd number (n, n+4), where n is an odd number. So, the first example yields 6. The second example, 6+9=15, takes the sum of the previous example, which is 6, and adds it to the next skipped odd number. So, if 5 is the previous odd number, 9 has to be the next odd number (5=n, therefore, n+4=9). So, it can be seen in these examples that the sum of the previous example is always added to the next odd number that occurs after skipping an odd number.

By looking at the sums of all these examples, there is a pattern that occurs. Each of these sums is every other odd triangular number, starting at the triangular number 6. So, 6 is the first triangular number, and 10 is the next, but it is not a sum of the second example, 6+9 because each sum of the products is every other triangular number. So, if 6 is the first triangular number seen here, 10 is skipped, 15 is the sum of the second example, and then 21 is skipped, and this pattern continues.

This can be expressed geometrically also by the following:





When taking the first triangular number, and the bottom two layers are added together, this gives us the odd number following the odd number skipped, and this is added to the sum of the previous summation. For example, the second triangle represents the triangular number 15. When the two bottom rows are added together, 5+4, this yields 9. 9 is the number that was added to the previous triangular number, which was 6, as can be seen. Therefore it works geometrically.