Kelsey Stavseth
Theresa Ryan
Dan Caffrey
Carrie Leonard

Final Draft 1.1

Can you figure out an efficient way to find triangular-square #'s? Do you think there are infinitely many?

We were first given that 1 and 36 are both triangular-square numbers. The first notable thing to try would be to set up Gauss's formula and see if you could derive squares from his formula for adding consecutive numbers.

Gauss's formula:

$\frac{n(n+1)}{2}$ → set equal to a square-triangular

- We already know (36): $36 = \frac{n(n+1)}{2} \rightarrow 72 = n(n+1)$

Here we know that $8 \cdot 9$ equals 72. However, setting each square number equal to $\frac{n(n+1)}{2}$ is extremely inefficient, considering that there are infinitely many square numbers.

We also tried adding up triangular numbers to find squares, and found this too was tedious:

1	+2	+3	+4	+5	+6	+7	+8	+9	+10	+11	+12
	3	6	10	15	21	28	36	45	55	66	

+13	+14	+15	+16	+17	+18	+19	+20	+21	+22	+23	+ - - -
78	91	105	120	136	153	181	200	220	241	263	285

Adding these numbers up hasn't found a square triangular number and to keep going would be inefficient, tedious

Kelsey Starseth
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and extremely frustrating! It is now obvious that using fairly elementary methods, square-triangular numbers are extremely difficult to find.

To aid our effort we used the internet and found a generating function on www.courseworkhelp.co.uk. This function, However, was extremely difficult to understand and basically useless due to our mathematical background.

However, this website did have some useful information on triangular-square numbers. The data gathered is as follows:

n	Triangular-squares	Square-root	original triangular
1	1	1	1
2	36	6	8
3	1225	35	49
4	41616	204	288
5	1413721	1189	1681
6	48024900	6930	9800
7	1631432881	40391	57121
8	55420693056	235416	332928
9	1882672131025	1372105	1940449
10	63955431761796	7997214	11309768
11	2172602007770041	46611179	65918161

These numbers are the first 11 square triangular and their square-roots as well as their original triangular number.

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These numbers did not yield a formula, that we could see, for generating square-triangular numbers. However, looking closely we found that there is a pattern between the square roots.

Taking the ratio of $\frac{n+1}{n}$ we found a number close to 6 (actual number tends to 5.82...) The data is as follows:

Square roots of triangular-squares	ratio
1	$\frac{6}{1} = 6$
6	$\frac{35}{6} = 5.8\overline{3}$
35	$\frac{205}{35} = 5.82857$
204	$\frac{1189}{204} = 5.82843$
1189	$\frac{6930}{1189} = 5.828427$
6930	$\frac{40391}{6930} = 5.828427$
40391	

This shows the ratios are tending to 5.828427.

This could possibly be a way to finding square-triangular numbers but has never been proved, as the decimal approximations are never completely accurate.

The data does however, appear to have a limit and perhaps one day can be proved.

However, today the square triangular numbers are still incomplete and with no exact way to measure them efficiently only conjectures and hypotheses exist.

Extention 1.1

After looking at triangular-square numbers and the ways they can be derived and studied we thought it would be interesting to explore something different but related. In our search we came across pentagonal numbers. Through some internet exploration we found that the first pentagonal numbers were 1, 5, 12 and 22. We also found a formula that would generate these numbers (mathworld.wolfram.com):

Pentagonal numbers:

$\frac{n(3n-1)}{2} \rightarrow$ gives any pentagonal number

A few simple examples are useful:

$n=1 \quad \frac{1(3(1)-2)}{2} = \frac{2}{2} = 1$ | $n=3 \quad \frac{3(3(3)-1)}{2} \rightarrow \frac{24}{2} = 12$

These show generally how the theorem works

The formula was difficult to show. We did not prove it, But we did take some cool observations from it. When compared to triangular numbers $\left(\frac{n(n+1)}{2}\right)$ we found that the only difference was between $(n+1)$ and $(3n-1)$. We found that the difference between the 2 grew by squares after $n=3$. Which was interesting

- Triangular numbers : 1, 2, 3, 6, 10, 15, 21
- Pentagolar numbers: 1 5 12 22 35 51 70
- Difference : 0 3 9 16 25 36 49

Difference between Difference: 6, 7, 9, 11, 13

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We also noted that the difference between the difference of the triangular numbers and the pentagonal numbers were getting bigger by 2 each time. This seems obvious now since 5 is 2 greater than 3. So this growth makes sense logically.

Another pattern we noticed was a second order recursion within the pentagonal numbers.

Again the pentagonal numbers increase:

n	PN	Difference
1	1	
		4
2	5	
		7
3	12	
		10
4	22	
		13
5	35	
		16
6	51	
.	.	.
.	.	.
.	.	.

It appears that the difference is increasing by three each time. So perhaps an equation can be formed.

Second order: $T_{n+1} = T_n + (A_n + 3)$

Where A_n is the difference between T_n and T_{n-1}

Example: $T_2 = T_2 + (A_2 + 3)$

$T_2 = 5 + (4 + 3)$

$= 5 + 7$

$T_2 = 12$ - next pentagonal number!

This appears to generate the pentagonal numbers.

This was interesting to find another recursion formula generating the same thing. It was also nice to see how a lot of patterns (such as triangular and pentagonal) are intertwined.