Characterizations of Unit Bar Visibility Graphs

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Joint work with …

- Natalia Veytsel, Skidmore ‘03
  - [DV] Unit bar-visibility graphs, Congressus Numerantium 160 (2003), 161-175.

- Ellen Gethner, CU Boulder

- Joan Hutchinson, Macalester College
  - [DGH2] A Characterization of Triangulated Polygons that are Unit Bar-Visibility Graphs, in preparation.
What is a Bar-Visibility Graph (BVG)?

- **Vertices** represented as horizontal **bars** in plane
- **Edges** correspond to unobstructed **visibility bands** between bars
  - Visibility bands must have **positive width**
Motivations and Applications

- Modeling tool for digital circuit design
- Design tool for diagrams to display relations among data
  - Bars or boxes can have data labels written in them
- Some visibility models can display non-planar graphs in the plane
Early Work on Bar-Visibility

- **1976, Garey, Johnson, and So**
  - Used visibility graph models to design a method of testing printed circuits
  - Reduced potential of $O(n^2)$ tests to maximum of 12.

- **1985, Schlag, Luccio, Maestrini, Lee, Wong**
  - Studied bar-visibility graphs as models of *stick diagrams* used in VLSI design

- **1985, Wismath ...**
- **1986, Tamassia and Tollis ...**
- **1986, Rosenstiehl and Tarjan**
  - Gave simple, linear-time characterization bar-visibility graphs in the plane
Bar-visibility graphs are fully characterized

- Theorem [TT, W, RT]. A graph $G$ is a BVG IFF it has a planar embedding with all cutpoints on a common face.
  - Necessity proof not hard:
    - Every BVG induces a planar graph with all cutpoints on outer face.
  - 2-connected sufficiency proof:
    - Uses $st$-numbering to construct layout
  - General sufficiency proof:
    - Not difficult to paste layouts of 2-connected blocks together.
- Corollary: Every planar graph is a subgraph of a bar-visibility graph.
What is a Unit Bar-Visibility Graph (UBVG)?

- It’s a BVG…
  - Vertices represented as horizontal bars in plane
  - Edges correspond to visibility bands between bars
- … in which all bars have equal length
Motivations and Applications

- **Circuit Design:**
  - Circuit components are often all same size

- **Display of Data:**
  - Same size bars permit readable type in all bars

- First step toward BVGs in which *ratio of bar-lengths is bounded below* by a constant
  - Ratio is one for UBVGs
What is and isn’t known about UBVGs

- Recall that for general BVGs …
  - Have linear-time characterization
  - Every planar graph is a subgraph of a BVG
- But for Unit Bar-visibility graphs …
  - No simple characterization known – no polynomial characterization known
  - Not every planar graph is a UBVG, or even a subgraph of a UBVG:
    - The complete graph $K_4$ cannot be a subgraph of a UBVG [DV]
UBVG results from [DV]

- $K_4, K_{2,3}$ are not UBVGs
- UBVG property is not hereditary, not preserved by contraction or subdivision (Pf: For $n \geq 6$, The circular ladder $K_2 \times C_n$ is UBVG iff $n$ is even)
- A tree is a UBVG if and only if it has maximum degree three and is a subdivided caterpillar.
Proof of UBVG Result for Trees

- **Sufficiency:**

- **Necessity proof idea:**
  - If vertex \( v \) in UBVG tree \( T \) has degree 3, then two neighbor bars protrude beyond \( b(v) \) and other does not (because no cycle).
  - If the three neighbors of vertex \( v \) also have degree 3, the above property forces a cycle.
  - In general, \( T \) cannot contain subdivision of this tree:
A graph $G$ is **outerplanar** if it has a plane embedding with all vertices on the outer face.

The **dual** $G^*$ of a plane graph $G$ has a vertex for each face and adjacency means two faces of $G$ share an edge.

The **weak dual** $G^*_w$ is the dual with the vertex representing the outer face removed.

**Proposition:** A plane graph $G$ is outerplane if and only if its weak dual $G^*_w$ is a tree.
Let $G$ be a UBVG, and $F \in G^*_{w^*}$.

- $\text{Rec}(F)$ is the rectangle bounding the bars representing $f$ in the UBV layout of $G$.
  - $F'$ is a **left-neighbor of $F$** if $\text{Rec}(F')$ protrudes to the left of $F$.
    - Analogous definition for **right-neighbor**.
  - $F'$ is a **down-neighbor of $F$** if it is not a left- or right-neighbor, and $\text{Rec}(F')$ protrudes below $F$.
    - Analogous definition for **up-neighbor**.
Theorem. If $G$ is an outerplanar UBVG, and $F \in G^*_w$, then

- $F$ has at most one left-, one right-, two up-, and two down-neighbors.
- If either $G$ is a near-triangulation or $G$ is triangle-free, then $F$ has at most one neighbor of each type.

Corollaries.

- If $G$ is a 2-connected, outerplanar, triangle-free UBVG, then $G^*_w$ is a subdivided caterpillar with maximum degree 4.
Theorem. [DV] If $G$ is a 2-connected, outerplanar graph such that $G^*_w$ is a path, then $G$ is a UBVG.
No triangulation with four or more vertices is a UBVG

Every noncollinear UBVG (i.e., no two bars have endpoint with equal x-coordinates) is a near-triangulation (all but external face are triangles).
Triangulated Polygons

- In [DGH], we characterize the **triangulated polygons** that are UBVGs.
  - Triangulated polygon = 2-connected near-triangulation
  - Vertices in $G^*_w$ have maximum degree *three*

- **Theorem.** If $G$ is a triangulated polygon that is a UBVG, then $G^*_w$ is a subdivided caterpillar with maximum degree 3.
Two Motivating Examples

- Triangulated polygon (TP) at right has UBVG layout:
  - Embedded in triangulated plane
  - Weak dual $G^*_w$ is subdivided caterpillar
  - The ‘spine’ generally moves left to right in triangulated plane, permitted also to go up or down, but not also right to left
  - The ‘legs’ go only up or only down from spine
- The 2nd TP is *not* a UBVG
The Internal Spine String
of a TP

- The *internal spine* $I$ is the sequence of faces in $G^*_w$ that lie strictly between the first and last degree-3 vertices (i.e., deg-3 faces of $G$)
  - $I$ may be empty
  - $I$ is uniquely determined by $G^*_w$
- Orient $I$ left-to-right, and label each face $F$ as follows:
  - $A$ (resp. $B$) if there is a leg incident *above* (below) $F$.
  - $N_A$ (resp. $N_B$) if $F$ borders infinite face *above* (below) $F$.
- Denote this string $S_I$ and call it the *internal spine string of* $G$. 
Spine String Examples

- Internal spine in blue
  - \( S_I = N_B N_A B N_A N_B \)
  - \( N_A N_B A A A N_B \)
  - \( B N_A A N_B N_A A B A B A N_A N_B A N_A B \)
We group each spine string into maximal substrings with only $A$ or $N_A$ terms \((A\text{-clumps})\) and only $B$ or $N_B$ terms \((B\text{-clumps})\).

- $S_I = B\ N_A\ A\ N_B\ N_A\ A\ B\ A\ B\ A\ N_A\ N_B\ A\ N_A\ B$
- $C(S_I) = C_1\ C_2\ C_3\ C_4\ C_5\ C_6\ C_7\ C_8\ C_9\ C_{10}\ C_{11}$

- $S_I = N_A\ N_B\ A\ A\ A\ N_B$
- $C(S_I) = C_1\ C_2\ C_3\ C_4$
Feasible Triangulated Polygons

- **Feasibility Theorem.** [DGH] Let \( G \) be a triangulated polygon whose weak dual \( G^*_w \) is a subdivided caterpillar with maximum degree 3. Let \( C(S_I) = C_1 \ldots C_k \) be the clumped internal spine string of \( G \). The following conditions are necessary for \( G \) to be a UBVG:
  1. Each clump \( C_i \) contains at most two A- or B-terms.
  2. If \( C_i \) contains the substring \( A (N_A)^s A \) or \( B (N_B)^s B \), then \( s = 0 \) or 1.

- A TP satisfying the conditions of the theorem is called feasible.
Consider the UBV layout of a single $A$-clump, with labels as shown.

- All $a$-bars must lie within the corridor shown of width $< 3$.
- Lemma. We may assume wlog that
  - All $c$-vertices lie within this corridor;
  - All $c$-vertices have disjoint corridors
  - An $a$-vertex between but not adjacent to two $c$-vertices has a disjoint corridor.
- Feasibility conditions follow.
Feasibility is Not Enough

- There are also **height** constraints on the $a$-bars of an $A$-clump.
- WLOG, we may assume that the sequence $\{y(a_i)\}$ in an $A$-clump may not decrease and then increase (no internal min height).
- Analogously the sequence $\{y(b_i)\}$ in a $B$-clump may not increase and then decrease (no internal max height).
- We can also assume that all **$c$-bars lies above all $a$-bars** (resp. $d$-bars below $b$-bars).
### Clump Types

<table>
<thead>
<tr>
<th>A-TYPE</th>
<th>B-TYPE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Max = ${N_A^{++}A N_A^{++},$</td>
<td>$\text{Min}_1 = {N_B^{++}B N_B^{++},$</td>
</tr>
<tr>
<td>$N_A^<em>A N_A^#A N_A^</em>}$</td>
<td>$N_B^<em>B N_B^#B N_B^</em>}$</td>
</tr>
<tr>
<td>$\text{MaxOrInc} = N_A^{++}A N_A^#$</td>
<td>$\text{MinOrDec} = N_B^{++}B N_B#$</td>
</tr>
<tr>
<td>$\text{MaxOrDec} = N_A^#A N_A^{++}$</td>
<td>$\text{MinOrInc} = N_B^#B N_B^{++}$</td>
</tr>
</tbody>
</table>

- Six types above are called **semirigid clumps**.

- **Wild**$_A = \{N_A^#A N_A^#, N_A^+\} \quad \text{Wild}_B = \{N_B^#B N_B^#, N_B^+\}$

- Four types above are called **wildcard clumps**. The wildcard clumps $S_A = A, N_A$, $S_B = B, N_B$ are **singleton wildcards**.
Given a feasible TP $G$, form its clumped internal spine string $C(S_I)$.

Find and label its semirigid clumps, $R_1, R_2, \ldots$.

A pair of two successive semirigid clumps, $(R_i, R_{i+1})$, is called a special needs pair if it is one of the following:

- **A.** (Max, Max), (Max, MaxOrDecrease), (MaxOrDecrease, Max), (MaxOrDecrease, MaxOrDecrease)
- **B.** (Min, Min), (Min, MinOrIncrease), (MinOrIncrease, Min), (MinOrIncrease, MinOrIncrease)
Example & Intuition

\[ C(S_1) = N_A^4 A N_A^2 N_B N_A^{100} B N_A^5 A \]

\[ = \text{Max } S_B \text{ Wild}_A S_B \text{ MaxOrInc} \]

\[ R_1 = \text{Max } \quad R_2 = \text{MaxOrInc} \]

\( (R_1, R_2) \) is a type-A special needs pair

The bars between a type-A special needs pair must form an internal min – by ‘calculus’

*In this example, do the in-between bars provide sufficient ‘wiggle room’ to permit a layout?*
Characterization Theorem

Theorem. [DGH] Let $G$ be a feasible TP. $G$ is a UBVG if and only if its internal spine string satisfies the following two conditions:

- **A.** If $G$ has a type-A special needs pair $(R_i, R_{i+1})$, then between $R_i$ and $R_{i+1}$ there must be at least one clump that is either (1) the singleton wildcard $N_A$, or (2) a non-singleton wildcard of type B (namely $N_B BN_B^\#$, $N_B^\# BN_B$, or $N_B^{++}$).

- **B.** If $G$ has a type-B special needs pair $(R_i, R_{i+1})$, then between $R_i$ and $R_{i+1}$ there must be at least one clump that is either (1) the singleton wildcard $N_B$, or (2) a non-singleton wildcard of type A (namely $N_A AN_A^\#$, $N_A^\# AN_A$, or $N_A^{++}$).
Show that a TP that violates the theorem cannot be laid out.

A violation of condition A means that between any type-A special needs pair \((R_i, R_{i+1})\):

- **Every** A-clump is a wildcard containing either an A-face or an \(N_A^2\) pair.
- **Every** B-clump is a singleton wildcard \(B\) or \(N_B\).

We contradict the assumption of a UBVG layout by:

- Starting at clump \(R_{i+1}\), show its first a-vertex protrudes left of its b-vertex
- Show that with intermediate clumps as above, this condition is maintained for all the intermediate A-clumps.
- Show that this implies that the first a-vertex is fully to the left of its b-vertex – contradicting that these vertices are adjacent.
Idea of Sufficiency Proof

- Assume the necessity conditions are satisfied.
- Step 1. Label each clump INC, DEC, INC-DEC, and DEC-INC.
  - Require successive clumps to have matching labels:
    - INC, INC-DEC, DEC, DEC-INC is OK
    - INC, INC, DEC is not OK
- Step 2. Using the INC/DEC labels, compute minimum overlap required for visibility;
- Step 3. Produce bar coordinates for layout
  - Can do this if G satisfies theorem conditions
Further Work and Open Questions

- Characterize all UBVG near-triangulations, i.e., do 1-connected case.
  - Not obvious that we can ‘paste’ together 2-connected layouts
- Are there other classes that can be characterized, beyond trees and triangulated polygons?
- Is recognition of UBVGs NP-complete?
  - Recognition of BVGs is linear, but recognition of Rectangle VGs is NPC (Shermer 1996).