Water Rockets in Flight: Calculus in Action

MATHEMATICS CLASSIFICATIONS:
Calculus

DISCIPLINARY CLASSIFICATIONS:
Physics (mechanics)

PREREQUISITE SKILLS:
1. Trigonometry
2. Vector calculus for three-dimensional space curves
3. Single-variable calculus for the simpler height vs. time model.

PHYSICAL CONCEPTS EXAMINED:
1. Orthogonality of speed and acceleration in horizontal and vertical directions
2. Relationship between position, velocity, and acceleration
3. Wind resistance

COMPUTING REQUIREMENT:
Computer algebra system or graphing calculator, for curve fitting
Abstract

We describe an easy and fun experiment using water rockets to demonstrate some of the concepts of multivariable calculus. After using video stills from a single water rocket launch to generate the raw data, we develop a model to analyze the rocket flight.

Because of factors such as rocket propulsion and wind effects, the water rocket does not follow the parabolic projectile trajectory commonly found in textbooks. Instead, we use polynomial interpolation to calculate the $X$, $Y$, and $Z$ coordinate functions of the rocket as a function of time during its entire flight. We then use methods from multivariable calculus to analyze the flight and to estimate quantities such as the maximum height reached by the rocket and curvature of the flight path that are not apparent from direct observation. Examination of first and second time derivatives of the rocket coordinates allows us to identify the thrusting, coasting, and recovery stages of the rocket flight, and comparison to the parabolic model shows the effects of the wind.

We offer two variations of the Module:

- One is very similar to that described above but uses a least-squares fit instead of polynomial interpolation to determine the coordinate functions.

- The other is a simpler model based on a one-variable polynomial fit giving the height of the rocket as a function of time, suitable for a first-semester calculus course.

The appendices include an optional overview of the curve-fitting techniques using linear algebra, a supplies list and procedure to launch and videotape a water rocket, an auxiliary set of video stills, and a complete Maple 8 code for generating the results.
1. Introduction

Water rockets are cheap, reusable, easy to launch, and have a very high fun-to-nuisance ratio. They also provide a simple example of some of the fundamental aspects of a model rocket flight. Because of factors such as rocket propulsion and wind effects (which can be classified as systematic if the wind is steady or as random if the wind is gusty), their flight paths are more complex than those of simple projectiles.

The goal of this Module is to model the flight path of a single water rocket launch and then use the tools of calculus to analyze the rocket’s performance. Most calculus textbooks include both one- and two-dimensional parabolic models describing the flight of a projectile. In the one-dimensional case, the function \( s(t) = vt - \frac{1}{2}gt^2 \) models the vertical position of a projectile with respect to time \( t \), where \( g (= 9.81 \text{ m/s}^2) \) is the gravitational constant and \( v \) is the initial velocity. In the two-dimensional case, the vector-valued function \( r(t) = (S \cos \theta)t, (S \sin \theta)t - \frac{1}{2}gt^2 \) models the planar trajectory of a projectile, where \( S \) is the initial speed and \( \theta \) is the initial launch angle as measured from the ground. In both cases, it is assumed that, other than the initial boost acceleration, gravity is the only force acting on the rocket (e.g., air resistance and Coriolis effects associated with the rotation of the Earth are ignored).

We originally wanted a simple and engaging experiment that would provide the raw data for using these models. However, we did not have ready access to a large windless space (such as a hangar) nor a mechanism to measure the initial velocity needed to illustrate these standard models.

Rather than trying to fit the experimental data to a parabolic model, we chose instead to generate and analyze a single data set and explore the vagaries introduced by wind and variable thrust. We then needed a suitable projectile. On the one hand, tossing a ball vertically into the air did not seem exciting enough. On the other hand, anything involving rocket propulsion by fuel combustion seemed a little too exciting and hence logistically too difficult; also, they move too fast for easy measurements and are too expensive. We wanted something cheap, easy, and safe. Water rockets are the perfect solution. Furthermore, their behavior is more varied than that of projectiles modeled by the parabolic functions. They have just enough complexity in their flight paths to provide an opportunity to put the skills and concepts learned in calculus to work in a substantial way.

A water rocket consists of a tapered plastic chamber about 13 cm long with small fins and a little hole in the base. The chamber is partially filled with water and then air is forced into the chamber with a manual air pump that clamps onto the base. This clamp is also the launch mechanism. See Figure 1.

When the rocket is released, the air and water escapes rapidly through the small hole in the base of the rocket, providing the power for the first stage of the rocket’s flight, the thrusting (or boost) stage. The rocket then continues to soar upwards, although more and more slowly, with now its acceleration only affected by gravity, air resistance, and (possibly) wind; this second stage of the
rocket’s flight is the *coasting* stage. The rocket then returns to the ground in the *recovery* stage (defined as the part of the rocket flight path from the time it reaches its maximum height until it lands).

This terminology is borrowed from more sophisticated rockets that often have a recovery mechanism (such as a parachute) to minimize damage to the rocket upon landing. Water rockets, however, do not need parachutes, since they are quite sturdy; and as long as they land on a soft surface such as turf, they will be undamaged.

Water rockets are quite lightweight, hence even a slight crosswind can significantly affect their flight paths. Indeed, only crosswind effects can cause the path of the rocket to exhibit nonplanar features, since gravity, thrust, and air resistance are all planar forces.

Because of the additional forces acting on a water rocket, simple projectile models are inadequate for analyzing its flight path. Excellent recent papers give well-developed models for the flight path of a generic water rocket; however, Prusa [2000] is beyond the scope of a typical undergraduate calculus sequence, and Finney [2000] considers only height vs. time rather than the three-dimensional space curve that we wanted to consider. Furthermore, since these models require ideal launch conditions (e.g., windless conditions and perfectly vertical launches) that are very difficult to achieve in practice, we chose instead to analyze a specific rocket-flight data set simply by fitting a smooth curve to a finite number of data points along the flight path.

To get the raw data for the experiment, we videotaped the flight of a water rocket, with a building of known height in the background. On the day of the launch, there was gusting wind of perhaps 8 to 24 km/h (5 to 15 mph); so naturally the rocket was blown off its planar (and parabolic) course. We noted the position of the rocket against the building in the video stills and then used...
building blueprints to measure the rocket’s horizontal and vertical positions with respect to the building. With basic trigonometry and the ground distances, we converted this information into estimates for the three-dimensional position of the rocket. Having found these coordinates, we then applied the curve-fitting capabilities of the computer algebra system Maple to construct a model of the flight path.

From the vantage point of a single video camera position, information concerning the depth position of the rocket is not available; so nonplanar crosswind effects are not directly observable through the present analysis. Since we expect wind gusts generically to possess planar components, wind effects are expected to be characterized by segments of the flight path with zero curvature (i.e., with parallel velocity and acceleration vectors).

With this model, we can use the tools of calculus to answer questions that could not be addressed just by watching the launch. For example, how high did the rocket go? How far did it travel? How sharp was its turn around? How long did the thrusting stage last? To what extent did wind effects modify the action of gravity? How far was the rocket blown off course by the wind? Answering these questions requires computing and analyzing the derivatives of the spatial coordinates (i.e., analyzing the velocity and acceleration vectors) and finding the Frenet-Serret curvature of the flight path (here, our model explicitly assumes a zero-torsion flight path). Although a planar parabolic-path model does not provide a good approximation of the rocket flight path, a comparison to our curve gives a good sense of the effects of the wind and thrust on the behavior of the rocket.

We use a sixth-degree polynomials in our model. An even-degree polynomial is appropriate for modeling the height of a rocket flight (why is an odd-degree polynomial unsuitable?). We found that second- and fourth-degree polynomials did not capture the wind effects and did not fit the data points well. Polynomials of degree greater than six fit the data points quite well (as might be expected) but were poor models for the flight; they varied too much laterally and “flattened out” at the top. Hence, they could not be used to get good estimates of the maximum height.

In the absence of air resistance and wind effects, there is no lateral acceleration, so the \( X(t) \) and \( Y(t) \) coordinate functions are expected to be linear in time \( t \). However, we did observe wind effects and hence we use sixth-degree polynomials for these coordinate functions too, to capture this phenomenon.

Experimentation with different curves can be quite instructive though, and we highly recommend doing it to see the effects of the various parameters on a model.

While most of this paper involves a three-dimensional space curve created with polynomial interpolation, we also include two possible ways of modifying the Module. One uses the least-squares fitting method instead of polynomial interpolation to fit the coordinate functions. Doing so gives slightly different answers in the analysis, but this version is not substantially different from the polynomial-interpolation model. The other variation is a simpler model, a
one-variable polynomial fit giving the height of the rocket as a function of time, suitable for a first-semester calculus course.

The remainder of this Module is organized as follows.

- **Section 2**: We provide a list of supplies needed to carry out this experiment and a description of the launch procedure.

- **Section 3**: We present the nine video stills used to generate the raw data for our model and introduce the ground and elevation diagrams needed to convert the apparent position of the rocket as observed from a fixed background.

- **Section 4**: We introduce the trigonometric formulas needed to convert the apparent position of the rocket into the three-dimensional coordinates $X(t)$, $Y(t)$, and $Z(t)$ as a function of time (as measured by the video camera). To generate smooth functions of time for the rocket coordinates, we use Maple 8 to compute the three components of the velocity and acceleration of the rocket (this Module could easily be adapted to any computer algebra system or graphing calculator with curve-fitting capabilities). From these components, we calculate the curvature of the rocket path as a function of time, which exhibits the expected peak near the turn-around point (when the rocket has reached its maximum height). Near the end of the rocket flight, we note an unusual feature in the graph of the curvature, which is explained by a strong gust of wind affecting the path of the rocket near the end of the flight.

- **Section 5**: We briefly discuss modifications of the curve-fitting model used in Section 4.

- **Section 6**: We comment on the validity of the zero-torsion model itself and suggest possible augmentation of this experiment.

- **Appendices**: We present sources for additional video stills, the Maple code used in the present work, an introduction to curve-fitting techniques for those familiar with basic linear algebra.

The annotated **References** include a few Internet resources for exploring the mathematics of model rockets in general.

## 2. Supplies and Procedure

### 2.1 Supplies Needed

- water rockets (obtain extra in case of defective rockets or cracking on landing); buy “water-powered rockets” from a local toy or hobby store, or order them on the Internet (see **Section 7.4**)
• a metric tape measure or a laser telemetry device (if available)
• blueprints for a nearby three-story building (if available—at least the basic dimensions of the building must be known)
• camcorder with videotape
• editing software to view the video frame by frame if available, or at least VCR with a pause button
• stopwatch if unable to view the video frame by frame

2.2 Launching Procedure

• Choose an appropriate backdrop, such as a building with known height. This project is more interesting if the building is only about three stories tall. The rocket will then appear to rise above the building, so that the maximum height has to be estimated by using calculus.

• Before launching the rocket, measure ground distances from the camcorder site to the launch site and from the launch site to the building. The camcorder and launch site should be in line with an easily identifiable location on the building. Measure the distance from the center of the camera lens to ground level. Check if “ground level” at the base of the camera and “ground level” at the base of the building are the same and adjust if necessary.

• Launch the rocket and record the flight on the camcorder. Preview a flight to be sure that the rocket is visible against the building in the video stills. A successful launch is one in which the rocket appears in front of the building both at the beginning and at the end of the flight and appears to rise above the roofline at its maximum height. Several launches may be necessary to achieve this result. Having someone say, for example, “This is the third trial” as you begin to record the launch will help identify the different trials when viewing the tape later.

• Once the rocket has landed, measure the ground distances from the camcorder to the landing site and from the launch site to the landing site.

• View the videotape of the successful launches, decide which one to model, and then gather at least nine data points on the path of the rocket, including the launching and landing points. Although an approximating curve could be determined from fewer points, more data gives greater flexibility in choosing which points to generate the curve in the case of the polynomial interpolation fit and a more accurate curve in the case of the least-squares fit. The blueprints of the building in the background will help determine the position of the rocket with respect to the building. If blueprints are unavailable, then make estimates based on the known height of the building and take measurements for the horizontal distances. If you are able to view the
tape frame by frame, the fact that most camcorders record at about 30 frames per second can be used to determine timing. Otherwise, use a stopwatch and the pause button to estimate as well as possible.

2.3 Notes on Measurements

• We measure distances in meters (m), time in seconds (s), and angles in radians.

• Measurement error can be significant in this experiment. If possible, take each measurement twice and average. With three significant digits (the likely limitation of measurements that students can make), each measurement has an implicit measurement uncertainty between 0.1% and 1%.

• The standard color video rate is 29.97 frames/sec, which we take to three significant digits as 30.0 frames per second [Compes 2003].

2.4 Recording Procedure

Once the experiment is completed and a launch chosen, view the videotape frame by frame and extract information about the apparent position of the water rocket during the entire flight path. Construct Table 1 with careful attention to proper labeling, significant digits, and units. Determine the entries for Table 2 and present them in a similar form. Develop a model for the rocket flight using either polynomial interpolation, least-squares fit, or the one-dimensional model. Analyze the model, being sure to address all the questions raised in Section 4.3.

3. The Water Rocket Flight

3.1 Building Blueprint and Video Stills

Figure 2 shows the nine video stills that were the raw data for the model. The rocket was launched from ground level, and the camcorder was 1.52 m above ground level.

Figure 3 shows the exterior blueprints of the library on our campus that we used as a backdrop for our experiment. The scale of the blueprints let us estimate the rocket’s horizontal and vertical position at various points during its flight. The initial and terminal positions of the rocket, as well as seven in-flight data points extracted from the video stills, are indicated in the figure.
0th (launch) frame or $t_1 = 0.000$ s.

7th frame or $t_2 = 0.233$ s.

11th frame or $t_3 = 0.367$ s.

15th frame or $t_4 = 0.500$ s.

50th frame or $t_5 = 1.67$ s.

55th launch frame or $t_6 = 1.83$ s.
59th launch frame or $t_1 = 1.920$ s.  

63rd launch frame or $t_1 = 2.10$ s.  

71st launch frame or $t_1 = 2.37$ s.  

**Figure 2.** Video stills from a successful launch.

### 3.2 Apparent Position of Data Points

The apparent locations of the rocket against the building from the point of view of the camcorder operator are listed in Table 1. The heights $V_i$ are measured up from ground level, and the horizontal distances $H_i$ are measured to the right of the point at the intersection of a perpendicular from the camcorder to the building (this point is marked $P$ in Figure 4). Also, the times $t_i$ are determined using a video rate of 30.0 frames/sec.

### 3.3 Ground and Elevation Diagrams

**Figure 4** is an aerial view of the rocket launch, showing the relative positions of the camcorder, building, and rocket launch and landing sites.

The data in Table 1 give only the apparent position of the rocket against the
Figure 3. Exterior blueprints of the library used as backdrop for the launch.
Table 1.
Observations.

<table>
<thead>
<tr>
<th>Frame</th>
<th>$t_i$ (s)</th>
<th>$H_i$ (m)</th>
<th>$V_i$ (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>7</td>
<td>2</td>
<td>0.233</td>
<td>0.610</td>
</tr>
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<td>11</td>
<td>3</td>
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<td>1.22</td>
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<tr>
<td>15</td>
<td>4</td>
<td>0.500</td>
<td>1.83</td>
</tr>
<tr>
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<td>5</td>
<td>1.67</td>
<td>8.53</td>
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</tr>
<tr>
<td>63</td>
<td>8</td>
<td>2.10</td>
<td>10.4</td>
</tr>
<tr>
<td>71</td>
<td>9</td>
<td>2.37</td>
<td>16.7</td>
</tr>
</tbody>
</table>

Figure 4. Ground diagram for the rocket launch, not to scale.
building. We use the similar triangles illustrated in Figure 5 to find the actual heights \(z_i\) of the rocket from the perceived heights \(V_i\) of the rocket against the building.

![Elevation diagram for the rocket launch, not to scale.](image)

**Figure 5.** Elevation diagram for the rocket launch, not to scale.

4. Developing and Analyzing the Model

4.1 Estimating Rocket Coordinates

We now use basic trigonometry to estimate the three-dimensional coordinates \((x_i, y_i, z_i)\) of the rocket from the perceived positions \((H_i, V_i)\) of the rocket relative to the building. Figure 6 illustrates the quantities that we must determine to approximate the \(x_i\) and \(y_i\) ground coordinates of the rocket at time \(t_i\). Note again that our zero-torsion model places the depth coordinate \(y\) on the straight line joining the launch site and the landing site.

Table 2 gives the quantities used and the coordinates calculated. We find the times \(t_i\) by dividing the frame number by 30.0 frames/sec. At the moment of launch \((t = 0)\), the horizontal and vertical distances are 0. Next, use the law of cosines to find angle \(A\), the angle at the camcorder between the rocket launch and landing sites:

\[
A = \arccos \left( \frac{(\text{dist}B)^2 + (\text{dist}C)^2 - (\text{dist}A)^2}{2 \times (\text{dist}B)(\text{dist}C)} \right) = 0.591 \text{ radians} = 67.7^\circ.
\]

Since the line from the camcorder through the launch site forms a right angle with the building, this angle can be used to find the horizontal distance \(H_0\) along the building at landing:
Figure 6. Location diagram for the rocket launch, not to scale.

Table 2. Intermediate values and estimated coordinates.

<table>
<thead>
<tr>
<th>(i)</th>
<th>(A_i) (radians)</th>
<th>(n_i) (m)</th>
<th>(m_i) (m)</th>
<th>(t_i) (s)</th>
<th>(x_i) (m)</th>
<th>(y_i) (m)</th>
<th>(z_i) (m)</th>
</tr>
</thead>
<tbody>
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<td>7.32</td>
<td>0.000</td>
<td>0.000</td>
<td>7.32</td>
<td>0.00</td>
</tr>
<tr>
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<td>7.02</td>
<td>0.233</td>
<td>0.172</td>
<td>7.02</td>
<td>2.64</td>
</tr>
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<td>0.367</td>
<td>0.331</td>
<td>6.75</td>
<td>3.68</td>
</tr>
<tr>
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<td>1.59</td>
<td>4.62</td>
<td>3.68</td>
</tr>
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<td>4.79</td>
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<td>1.66</td>
<td>4.50</td>
<td>3.07</td>
</tr>
<tr>
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<td>1.97</td>
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<td>26.9</td>
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<td>1.79</td>
<td>4.28</td>
<td>2.05</td>
</tr>
<tr>
<td>9</td>
<td>0.591</td>
<td>29.9</td>
<td>4.11</td>
<td>2.37</td>
<td>2.29</td>
<td>3.42</td>
<td>0.00</td>
</tr>
</tbody>
</table>
\[ H_9 = (BuildingDistance) \tan A = 16.7 \text{ m}. \]

Use the horizontal distances \( H_i \) from Table 1 to find the ground angles between the launch site and the rocket position at each time (see Figure 6). To find the other angles listed in Table 2, use

\[ A_i = \arctan \frac{H_i}{BuildingDistance} \text{ radians}. \]

Also, use \( A \) and the fact that the line from the camcorder through the launch site to the building is perpendicular to the building to find the hypotenuses \( n_i \) (see Figure 6):

\[ n_i = \frac{BuildingDistance}{\cos A_i}. \]

Next, use the law of cosines to find \( B \), the angle at the launch site from the camcorder to the landing site:

\[ B = \arccos \left( \frac{(distB)^2 + (distC)^2 - (distA)^2}{2 \times (distB)(distC)} \right) = 0.532 \text{ radians} = 60.9^\circ. \]

Use the proportions from the law of sines to find the lengths of \( m_i \) (see Figure 6):

\[ m_i = \frac{(distC) \sin B}{\sin(\pi - (B + A_i))}. \]

Assuming that the rocket’s path remains in the vertical plane containing its launch and landing sites (i.e., the zero-torsion model), the \( A_i \) and \( m_i \) values give the angles and magnitudes for the projection into the \( X-Y \) plane of the position vectors for the rocket at time \( t_i \):

\[ x_i = m_i \cos \left( \frac{\pi}{2} - A_i \right), \quad y_i = m_i \sin \left( \frac{\pi}{2} - A_i \right). \]

Now, use similar triangles to approximate the actual rocket heights from the perceived heights against the building (see Figure 5). First, define \( z_1 = 0 \) and \( z_9 = 0 \) as the launch and landing heights. Next, estimate the remaining actual heights using similar triangles. Remember to subtract the camcorder height from the \( V_i \) and then add it back in to get the actual rocket heights \( z_i \). Thus, we get:

\[ z_i = CamHeight + \frac{(V_i - CamHeight)(m_i)}{n_i}. \]

We plot the resulting points \((x_i, y_i, z_i)\) in three-dimensional space in Figure 7.
4.2 Modeling the Flight Path Using Polynomial Interpolation

With estimates of the three-dimensional coordinates for the rocket at nine different times during its flight, we can create a space curve modeling the flight path. We use seven of the nine data points (omitting the in-flight points at times $t_3$ and $t_6$) to create a sixth-degree interpolating polynomial in the $X$, $Y$, and $Z$ components. We use seven points, because to determine all of the coefficients (including the constant term) uniquely, the number of points must be exactly one more than the degree of the polynomial. We omitted the third and sixth data points after experimenting with omitting different pairs of points to see which yielded the best model. In the absence of more sophisticated curve fitting techniques, this experimentation is an important part of the modeling process.

The resulting sixth-degree interpolating polynomials obtained from our zero-torsion model for the $X$, $Y$, and $Z$ coordinates are:

$$X(t) = 0.0412t + 4.65t^2 - 8.7t^3 + 7.63t^4 - 3.12t^5 + 0.480t^6,$$

$$Y(t) = 7.32 - 0.0701t - 7.90t^2 + 14.8t^3 - 13.0t^4 + 5.31t^5 - 0.816t^6,$$

$$Z(t) = 15.0t - 20.8t^2 + 27.3t^3 - 24.3t^4 + 10.3t^5 - 1.59t^6.$$

We graph them separately in Figure 8 and together as a parametrized curve in three-space in Figure 9.

As in each separate coordinate, the interpolating curve fits all of the data points well, even the points corresponding to times $t_3$ and $t_6$. The coefficients of the sixth-degree interpolating polynomials are not required to possess physical interpretations, in contrast to, for example, a least-squares fit of the flight path based on a parabolic model.

The polynomial fit curve shows the effect of the gusting wind on the empty (and hence light) rocket toward the end of its flight. It also gives a means of
estimating the maximum height attained by the rocket, a quantity which could not be determined from the raw data alone. This and other applications of the model are addressed in next section.

4.3 Analysis of the Flight

Many questions naturally arise about the rocket flight path:

1. How high did the rocket go?
2. How fast was it going at different times?
3. How much of that speed was due to vertical motion, and how much to lateral motion?
4. What was its acceleration at different times?
5. How far did the rocket travel during its flight?
6. How sharply did it turn around when it reached its apex? Was that the tightest turn it made during the flight?
7. When did the three stages (thrusting, coasting, recovery) of the flight occur?
8. To what extent did the wind counteract gravity?
9. How much did the wind blow the rocket off the parabolic path predicted by the standard model?

From just the video stills and the resulting data, we cannot address any of these questions. However, now that we have a model approximating the rocket flight, we can apply the calculus to find reasonable solutions for these questions.

The analysis below illustrates the ways that our curve may be physically interpreted. Another rocket flight will undoubtedly lead to a different-shaped curve with different physical properties. For example, in response to question 6, the graph shows that a wind gust with considerable horizontal strength actually lifted the rocket toward the end of its flight. A significantly different curve may result from other wind effects, or a windless day, and consequently must be considered in its own right rather than just mimicking our analysis.

To address these questions, we use the space curve \( \mathbf{R}(t) \) modeling the rocket flight that we found using sixth-degree polynomial interpolation:

\[
\mathbf{R}(t) = (X(t), Y(t), Z(t)).
\]

We also need the velocity vector \( \mathbf{R}'(t) \) (with units of m/s) and acceleration vector \( \mathbf{R}''(t) \) (with units of m/s\(^2\)), vector-valued curves that can be found by taking the component-wise first and second directives of \( \mathbf{R}(t) \), respectively:

\[
\mathbf{R}'(t) = (X'(t), Y'(t), Z'(t)), \quad \mathbf{R}''(t) = (X''(t), Y''(t), Z''(t)).
\]

Specifically, we have

\[
\begin{align*}
X(t) &= 0.0412 + 9.29t - 26.2t^2 + 30.5t^3 - 15.6t^4 + 2.88t^5, \\
Y(t) &= -0.07017 - 15.8t + 44.5t^2 - 51.9t^3 + 26.6t^4 - 4.90t^5, \\
Z(t) &= 15.0 - 41.6t - 81.8t^2 + 97.4t^3 + 51.4t^4 - 9.57t^5.
\end{align*}
\]

and

\[
\begin{align*}
X(t) &= 9.29 - 52.4t + 91.6t^2 - 62.5t^3 + 14.4t^4, \\
Y(t) &= -15.8 + 89.0t - 156t^2 + 106t^3 - 24.5t^4, \\
Z(t) &= -41.6 + 164t - 292t^2 + 206t^3 - 47.8t^4.
\end{align*}
\]

We use the velocity and acceleration vectors also to calculate the Frenet-Serret curvature for the three-dimensional parametrized space curve.

**Requirement 1: How high did the rocket go?**

This is perhaps the most natural question to ask that cannot be readily answered simply by watching the videotape. However, now that we have a smooth flight-path curve, finding the maximum height of the rocket is easily done by setting the vertical component of the velocity equal to zero and solving
for $t = t_{\text{max}}$, the time when the rocket reached its apex. We then find the maximum height by substituting $t_{\text{max}}$ into $Z(t)$:

$$Z'(t) = 15.0 - 41.6t + 81.8t^2 - 97.4t^3 + 51.4t^4 - 9.57t^5;$$

$$Z'(t) = 0 \text{ when } t = 0.974 \text{ and } Z(0.974) = 5.81 \text{ m.}$$

A maximum height of about 5.81 m is not bad for a water-propelled rocket about 13 cm long! However, in the absence of air resistance and with an initial vertical speed of 15.0 m/s, the rocket should have reached a maximum height of $(15.0)^2/(19.6) = 11.5 \text{ m}$, or almost twice as high as our model predicts. One can readily see the effect of air resistance on the maximum height.

**Requirement 2: How fast was the rocket going at the launch, apex, and landing times? How much of that speed was due to vertical motion, and how much to lateral motion? What was its acceleration at those times?**

To address these questions, we find and analyze the velocity and acceleration vectors and their magnitudes. We begin first with the components of the velocity vectors, with each component measured in m/s:

$$\mathbf{R}'(0) = (0.0412, -0.0701, 15.0),$$

$$\mathbf{R}'(t_{\text{max}}) = (0.947, -1.59, 0),$$

$$\mathbf{R}'(t_{\text{land}}) = (3.88, -6.59, -13.5).$$

Here $t_{\text{land}} = t_9 = 2.37 \text{ s}$. The signs of the components of each velocity vector indicate direction, information that is lost when we compute the overall speed below. At the launch, the derivatives in the $X$ and $Y$ components are very small (in absolute value), reflecting the fact that we actually did a pretty good job of launching the rocket vertically (i.e., the initial launch angle is $\theta = 89.7^\circ$ as measured from the horizontal), and that the later lateral motion must have been due to the wind. At the apex of the flight path, even though the $Z(t)$ derivative is zero, the $X(t)$ and $Y(t)$ derivatives indicate that the rocket was still moving laterally. Upon impact, although the rocket hit the ground with nearly the same vertical velocity as it left the ground, it was still moving quite fast laterally.

To see the overall speed at these three times, we need the magnitudes of the velocity vectors. We find that their magnitudes are

$$\text{speed}(0) = |\mathbf{R}'(0)| = 15.0 \text{ m/s} = 33.5 \text{ mph},$$

$$\text{speed}(t_{\text{max}}) = |\mathbf{R}'(t_{\text{max}})| = 1.85 \text{ m/s} = 4.14 \text{ mph},$$

$$\text{speed}(t_{\text{land}}) = |\mathbf{R}'(t_{\text{land}})| = 15.5 \text{ m/s} = 34.6 \text{ mph},$$

where we use the conversion $1 \text{ mph} = 0.447 \text{ m/sec}$.

Comparing these speeds to the values of the individual components shows that the speed at launch was almost entirely in the vertical direction, whereas when the rocket landed, because of the lateral motion, it was moving faster.
Now, consider the acceleration vectors at these three times, with each component measured in m/s$^2$:

\[
\mathbf{R}''(0) = \langle 9.29, -15.8, -41.6 \rangle,
\]
\[
\mathbf{R}''(t_{\text{max}}) = \langle 0.418, -0.711, -12.4 \rangle,
\]
\[
\mathbf{R}''(t_{\text{land}}) = \langle 21.8, -37.1, -65.3 \rangle.
\]

The negative value of the acceleration in each $Z$ component illustrates the general property that when an object is slowing down, the acceleration is in the opposite direction of the motion, and when it is speeding up, the acceleration is in the same direction as the motion. Thus, when rising and slowing down, the rocket has negative acceleration (in opposite direction to its ascent), and when falling and speeding up, the rocket also has negative acceleration (in the same direction as its descent).

Next, we find the lengths of the acceleration vectors at these three times, each again measured in m/s$^2$. We can then compare these values to some common known accelerations.

\[
\text{acceleration}(0) = |\mathbf{R}''(0)| = 45.4,
\]
\[
\text{acceleration}(t_{\text{max}}) = |\mathbf{R}''(t_{\text{max}})| = 12.4,
\]
\[
\text{acceleration}(t_{\text{land}}) = |\mathbf{R}''(t_{\text{land}})| = 78.2.
\]

The starting acceleration is about 50% more than that of a Space Shuttle at takeoff (29 m/s$^2$), the apex acceleration is about 50% more than that of a cheetah at takeoff (7.8 m/s$^2$), and the landing acceleration is about that of a parachute at landing (35 m/s$^2$) [Vawter 2003]. We quickly point out, however, that although the acceleration at apex is relatively close to the theoretical prediction of 9.8 m/s$^2$ for $g$, the landing acceleration indicates that the rocket hit the ground with an acceleration of about $8g$, which is physically impossible!

Considering the effects of air resistance and gravity alone, we conclude on physical grounds that the magnitude of the vertical component of the acceleration should be larger than the gravitational acceleration ($g = 9.81$ m/s$^2$) during ascent (when gravity and air resistance are in the same direction), while it should be less than the gravitational acceleration during descent (when gravity and air resistance are in opposite directions). Although the numerical results for the acceleration show a vertical component relatively close to $g$ at the apex, the values at launch and landing times raise serious doubts about the numerical validity of the model.

**Requirement 3.** How far did the rocket travel during its flight? Looking at the flight path, it seems that the rocket went farther coming down than going up, but how much further?

We can answer this question by finding and interpreting the arclength of the flight path, over various time intervals. First, we determine the length over
the entire flight:

\[
\int_0^{t_{\text{land}}} |R'(t)| \, dt = \int_0^{t_{\text{land}}} \sqrt{[X'(t)]^2 + [Y'(t)]^2 + [Z'(t)]^2} \, dt = 12.8 \text{ m}.
\]

Then, we determine the lengths over the rising and falling parts of the flight:

\[
\int_0^{t_{\text{max}}} |R'(t)| \, dt = \int_0^{t_{\text{land}}} \sqrt{[X'(t)]^2 + [Y'(t)]^2 + [Z'(t)]^2} \, dt = 6.26 \text{ m},
\]

\[
\int_{t_{\text{max}}}^{t_{\text{land}}} |R'(t)| \, dt = \int_{t_{\text{max}}}^{t_{\text{land}}} \sqrt{[X'(t)]^2 + [Y'(t)]^2 + [Z'(t)]^2} \, dt = 6.54 \text{ m}.
\]

The rocket traveled about 12.8 m, about 6.26 m during the first part of the flight and 6.54 m during the second part (when the rocket was lighter and the wind blew it more off course). In fact, taking into account the effects of air resistance and gravity alone, we would expect on physical grounds that the distance covered during descent should be shorter than the distance covered during ascent. Hence, the longer distance during the descent is indeed an indication of a wind gust with significant in-plane strength.

**Requirement 4. How sharply did the rocket turn around when it reached its apex? Is this the tightest turn it made during its flight?**

The Frenet-Serret curvature function, \( \kappa(t) \), helps answer this question. Recall that the radius of the osculating circle is the reciprocal of \( \kappa \); so the bigger \( \kappa \) is, the tighter the curve:

\[
\text{curvature} = \kappa(t) = \frac{|R'(t) \times R''(t)|}{|R'(t)|^3}.
\]

The curvature at the apex, when \( t = t_{\text{max}} = 0.974 \text{ s} \), is \( \kappa t_{\text{max}} = 3.62 \text{ m}^{-1} \). Thus, the osculating circle has radius \( 1/\kappa t_{\text{max}} = 0.276 \text{ m} \), which is about the size of an extra large pizza.

But is this necessarily the maximum curvature? Consider the graph of the curvature (Figure 10).

Since \( \kappa'(t) = 0 \) when \( t = 0.963 \text{ s} \), the maximum curvature is \( \kappa(0.963) = 3.65 \text{ m}^{-1} \), with osculating circle radius of 0.276 m. Therefore, it looks as if the maximum curvature occurred slightly after the rocket turned around at its apex. Since the values are so close, it is difficult to determine if this is due to the wind pushing sideways as the rocket slowed, and thus “loosening” the curve, or whether it is due to the vagaries of the model.

We notice a remarkable feature of curvature versus time near the 1.9-s mark. There the curvature becomes very small, indicating that the velocity and acceleration become nearly parallel for a short time. This is possible if a wind gust with significant planar strength appeared at that time.
Requirement 5. When did the three stages of thrusting, coasting, and recovery of the rocket flight occur?

While it would be quite difficult to determine the answer by watching the videotape (especially the transition from thrusting to coasting), and almost impossible to determine it from the nine data points, the stages can be quite clearly seen from the plots of the magnitudes of the velocity and acceleration functions.

First, we plot in Figure 11 the three components of the velocity curve.

![Figure 11](image)

Then we find the speed (magnitude of the velocity) and plot the resulting curve (Figure 12).

\[
\text{speed}(t) = |R'(t)| = \sqrt{[X'(t)]^2 + [Y'(t)]^2 + [Z'(t)]^2}.
\]

The speed curve gives a preliminary indication of the stages. We already know the transition point between the coasting and recovery stages, since that is just the time when the rocket reached its maximum height at \( t_{\text{max}} = 0.974 \) s.

The more subtle question is the transition between the thrusting and coasting stages. Note that the rocket starts out going quite fast, and then the speed decreases. There is a slight bend in the speed curve around 0.3 seconds, but it
is hard to see exactly what is happening when. The magnitude of the acceleration will give us more information. First, we plot the three components of the acceleration curve (Figure 13).

Then we find and plot the acceleration magnitude curve (Figure 14).

\[
|\text{acceleration}(t)| = | \mathbf{R}''(t)| = \sqrt{[X''(t)]^2 + [Y''(t)]^2 + [Z''(t)]^2}.
\]
We can see the transition from the thrusting stage to the coasting stage more clearly in the acceleration magnitude graph. The rocket was accelerating (although less and less as it lost water and pressure) from launching to somewhere around 0.5 s and then leveled off somewhat during the coasting stage. More specifically, we have

\[ \frac{d}{dt} (\text{speed}(t)) = \frac{d}{dt} |\mathbf{R}'(t)| = 0 \text{ when } t = 0.518 \text{ s.} \]

Between 1.8 s and 2 s, apparently a strong gust of wind caught the rocket and forced it sideways and down. The end of the coasting stage is when the rocket turns around at \( t_{\text{max}} \), and the rest of the flight is the recovery.

Therefore, the thrusting went from \( t = 0 \) s to \( t = 0.518 \) s, the coasting from \( t = 0.518 \) s to \( t_{\text{max}} \), and the recovery from \( t_{\text{max}} \) to \( t_{\text{land}} \).

**Requirement 6. To what extent did the wind counteract gravity?**

Since gravity is in the vertical direction, it suffices to consider just the vertical component of the acceleration vector, and we need to compare this to the constant acceleration due to gravity (\( -9.81 \text{ m/s}^2 \)), which otherwise would be the only force acting on the rocket after the thrusting stage finished.

In the absence of wind effects or air resistance, we would have expected \( Z''(t) \) to increase to \(-9.81 \text{ m/s}^2 \) during the thrusting stage and then remain constant for the rest of the flight. Any deviation from this thus must reflect the effects of wind gusts and/or air resistance. **Figure 14** indicates that the upward force from the wind actually exceeded the downward force of gravity.

Setting \( Z''(t) = 0 \), we find that \( t = 1.57 \) s to \( t = 1.90 \) s was the interval during which this occurred. Also, toward the end of the flight, a gust of wind actually pushed the rocket fairly hard into the ground. However, there was comparatively little wind effect during the coasting stage, as evidenced by
the curve staying fairly close to the gravitational constant. Also, compare the
two times when \(Z''(t)\) is zero to approximately the same times of the two
"dips", the second one quite sharp, in the previous acceleration magnitude
graph (Figure 13).

**Requirement 7. How much was the rocket blown off course by the wind?**

One way of getting a sense of this is to compare our model to the parameterized curve \(\langle (S \cos \theta)t, (S \sin \theta)t - \frac{1}{2}(9.81)t^2 \rangle\), which gives the trajectory of a projectile with no other forces acting on it except gravity. Here \(S\) is the initial speed, \(\theta\) is the launch angle, \(g\) is the gravitational constant \((-9.81 \text{ m/s}^2\)), and the initial position at \(t = 0\) is \((x, z) = (0, 0)\). Of course, this does not model our thrusting stage, but it is the best general model that we have.

Since the standard model is in two dimensions, we first need to reparametrize our model along the ground path from launch to landing to get a two-dimensional curve. Thus, the first coordinate function is
\[
S = \sqrt{(X(t_2) - X(0))^2 + (Y(t_2) - Y(0))^2 + (Z(t_2) - Z(0))^2}
\]
and the second coordinate function is simply \(Z(t)\).

To use the standard model, we need values for the initial speed \(S\) and launch angle \(\theta\). We start by estimating them from the \(t_1\) and \(t_2\) data points. Then \(S\) is just the change in distance divided by the change in time, and \(\theta\) is the launch angle from the launch point at \(t = t_1 = 0\) to the position at \(t = t_2\):
\[
S = \frac{\sqrt{(X(t_2) - X(t_1))^2 + (Y(t_2) - Y(t_1))^2 + (Z(t_2) - Z(t_1))^2}}{t_2 - t_1} = 11.4 \text{ m/s}^2,
\]
\[
\theta = \arccos \left( \frac{\sqrt{(X(t_2) - X(t_1))^2 + (Y(t_2) - Y(t_1))^2}}{\sqrt{(X(t_2) - X(t_1))^2 + (Y(t_2) - Y(t_1))^2 + (Z(t_2) - Z(t_1))^2}} \right)
\]
\(= 1.44 \text{ radians}.\)

The initial velocity is less than the "instantaneous" initial speed of 15.0 m/s found in **Requirement 2** and reflects the changing acceleration during the thrusting stage.

We find the appropriate \(t\) ranges by assuming that the second coordinate remains positive. Thus, we solve:
\[
(S \sin \theta)t - \frac{1}{2}(9.81)t^2 = 0.
\]
The two solutions are
\[
t = \text{standard launching time} = 0.000 \text{ s}, \quad t = \text{standard landing time} = 2.31 \text{ s}.
\]

The difference in the landing positions gives us a sense of how far "off course" the rocket was blown:
\[
\text{landing difference} = \text{DistA} - (S \cos \theta) \times \text{(standard landing time)}
\]
\(= 4.52 - 3.37 = 1.15 \text{ m}.
\]
Without wind, the standard model estimates that the landing site would have been about 3.37 m away from the launching site, but the rocket actually landed 4.52 m away, a difference of 1.15 m.

We plot and compare our model to the standard parabolic model (Figure 16).

It might be argued that the launch angle was actually directly up and so the vertical deviation was due to wind. Furthermore, the rocket was accelerating from $t_1$ to $t_2$, so the $S$ and $\theta$ values used above may not be the best parameter values for a good comparison. If the initial value shifts at $t = t_i$ to $(x_i, z_i)$, the model becomes

$$\langle (S \cos \theta)(t - t_i) + x_i, \ (S \sin \theta)(t - t_i) - \frac{1}{2}gt(t - t_i)^2 + z_i \rangle.$$

We estimated the parameters $S$ and $\theta$ using times $t_2$ and $t_3$, since the thrust was diminished by then. Figure 17 gives the plot of our model vs. the adjusted standard model in Figure 16.

The figure illustrates that there was a thrusting phase occurring during the actual rocket flight. Also, while this two-dimensional model involves no wind effect, it gives a greater ground distance than was actually covered by the rocket. Since the rocket was in fact blown by the wind, this is probably a less useful model than the parabolic model derived using $t_1$ to $t_2$.

5. Model Variations

5.1 Modeling the Flight Path Using Least-Squares

We can also find a viable model through a least-squares fit of all nine data points of Table 1. The model created using a sixth-degree least-squares fit is quite similar in shape to the sixth-degree polynomial interpolation of Section 4.2. For example, both models achieve similar maximum height. Preference for one method over the other may depend upon which modeling tools are available, where this Module is integrated in the curriculum, and mathematical background.

Using the times the times $t_i$ and coordinates $x_i, y_i,$ and $z_i$ from Table 2, we can find a least-squares fit for the $X, Y,$ and $Z$ components. The resulting sixth-degree least-squares polynomials for the $X, Y,$ and $Z$ components, measured in meters, are:

$$X(t) = 0.161t + 3.80t^2 - 6.74t^3 + 5.77t^4 - 2.37t^5 + 20.369t^6,$$
$$Y(t) = 7.32 - 0.277t - 6.46t^2 + 11.5t^3 - 9.81t^4 + 4.03t^5 - 0.627t^6,$$
$$Z(t) = 14.6t - 18.1t^2 + 20.7t^3 - 18.1t^4 + 7.74t^5 - 1.22t^6.$$

We can then combine the least-squares fits in the three components to create a space curve to model the rocket’s flight (Figure 18).

As with the interpolated version, this curve fits the data points well and can be used to analyze the rocket flight. It gives slightly different values for the
Figure 16. Actual flight path (solid line) vs. standard model (dashed model).

Figure 17. Actual flight path (solid model) vs. adjusted standard model (dashed line).

Figure 18. Space curve model of the rocket’s flight from sixth-degree least-squares fit to the data.
questions considered in Section 4.3. For example, to estimate the maximum height, we compute the derivative of \( Z(t) \), and find the critical value of \( Z(t) \):

\[
Z'(t) = 14.6 - 36.2t + 62.0t^2 - 72.4t^3 + 38.7t^4 - 7.32t^5,
\]

\( Z'(t) = 0 \) when \( t = 0.958 \) s and \( Z(0.958) = 5.68 \) m.

Thus, according to our least-squares curve, the rocket reached its maximum height approximately 0.958 s into the flight and the height was approximately 5.68 m, which is less than the 5.81 m approximation found with the interpolated curve in Section 4.3.

### 5.2 Simpler Height vs. Time Model

It is also possible to use calculus of one variable to analyze the rocket flight from the height function alone. Much of the work is very similar to that in the three-dimensional case: Estimating the actual rocket positions is done as in Section 4.1, and finding a model \( Z(t) \) for the height as a function of time uses interpolation as in Section 4.2. The functions \( X(t) \) and \( Y(t) \) do not need to be determined. Figure 19 shows the data points and the curve \( Z(t) \).

Since the height function \( Z(t) \) is identical to the third (vertical) component of the three-dimensional model, some of the analysis below repeats what was presented earlier. We include this as necessary to provide a complete consideration of the height vs. time model.

The height function alone does not model the strong lateral drift due to the wind at the end of the flight, but it does show the slight lift and sharp drop at the end of the flight as the wind forced the rocket down.

Figure 19 shows the data points and the curve \( Z(t) \).

![Graph of Z(t)](image)

**Figure 19.** Model of the rocket’s height from sixth-degree least-squares fit to the data.

As in Section 4.3, we can address some questions about the flight, using the first and second derivatives of \( Z(t) \):

\[
Z'(t) = 15.0 - 41.6t + 81.8t^2 - 97.4t^3 + 51.4t^4 - 9.57t^5,
\]

\[
Z''(t) = -41.6 + 164t - 292t^2 + 206t^3 - 47.8t^4.
\]
Requirement 1. How high did the rocket go?

Setting $Z'(t) = 0$ and solving the time $t_{\text{max}}$ when the rocket reached its apex. We find the maximum height by substituting $t_{\text{max}}$ into $Z(t)$:

$$Z'(t) = 0 \text{ when } t = t_{\text{max}} = 0.974 \text{ s and } Z(0.974) = 5.81 \text{ m.}$$

Requirement 2. How fast was the rocket going at the launch, apex and landing times? What was its acceleration at those times?

To find the speeds, we substitute the three times into $Z'(t)$:

$$\text{speed}(0) = |R'(0)| = 15.0 \text{ m/s} = 33.5 \text{ mph},$$
$$\text{speed}(t_{\text{max}}) = |R'(t_{\text{max}})| = 1.85 \text{ m/s} = 4.14 \text{ mph},$$
$$\text{speed}(t_{\text{land}}) = |R'(t_{\text{land}})| = 15.5 \text{ m/s} = 34.6 \text{ mph},$$

$$Z'(0) = 15.0 \text{ m/s} = 33.5 \text{ mph},$$
$$Z'(t_{\text{max}}) = 0.0 \text{ m/s} = 0.0 \text{ mph},$$
$$Z'(t_{\text{land}}) = -13.5 \text{ m/s} = -30 \text{ mph.}$$

Next, we find the accelerations at these three times, each measured in m/s$^2$:

$$Z''(0) = -41.6,$$
$$Z''(t_{\text{max}}) = -12.4,$$
$$Z''(t_{\text{land}}) = -65.3.$$

Requirement 3. When did the three stages, thrusting, coasting and recovery, of the rocket flight occur?

The recovery stage begins at time $t_{\text{max}}$. The end of the thrusting stage (and beginning of the coasting stage) can be more subtle. We first consider the graphs of the velocity $Z'(t)$ and the acceleration $Z''(t)$ functions, plotted together in Figure 20.

The thrusting stage ends with the first maximum of $Z''(t)$, so we must determine when this occurs:

$$\frac{d}{dt} Z''(t) = 0 \text{ when } t = 0.517 \text{ s.}$$

Therefore, the thrusting went from $t = 0 \text{ s}$ to $t = 0.517 \text{ s}$, the coasting from $t = 0.517 \text{ s}$ to $t = t_{\text{max}}$, and the recovery from $t = t_{\text{max}}$ to $t = t_{\text{land}}$. Between 1.8 s and 2 s, apparently a strong gust of wind caught the rocket and forced it sideways and down.
Requirement 4. To what extent did the wind counteract gravity?

Since gravity is in the vertical direction, it suffices to consider just the vertical component of the acceleration vector, and we need to compare this to the constant acceleration due to gravity ($-9.81 \text{ m/s}^2$), which otherwise would be the only force acting on the rocket after the thrusting stage finished.

If there had been no wind or air resistance, we would have expected $Z''(t)$ to increase to $-9.81 \text{ m/s}^2$ during the thrusting stage and then remain constant for the rest of the flight. Any deviation from this thus must reflect these effects. The graph of $Z''(t)$ in Figure 20 (solid curve) indicates that the upward force from the wind actually exceeded the downward force of gravity. Setting $Z''(t) = 0$, we find that $t = 1.55 \text{ s}$ to $t = 1.95 \text{ s}$ seconds was the interval during which this occurred. Also, toward the end of the flight, a gust of wind actually pushed the rocket fairly hard into the ground. However, there was comparatively little wind effect during the coasting stage as evidenced by the curve staying fairly close to the gravitational constant.

Requirement 5. What effect did the wind have on the height attained by the rocket?

We use the standard parabolic model for the height of a projectile, $s(t) = v_0 t - \frac{1}{2} gt^2$, to get some sense of what effect the wind might have had on the height.

We need a value for the initial velocity $v_0$. We start by estimating it from the $t_1$ and $t_2$ data points. The velocity $v_0$ is then approximately the change in distance divided by the change in time:

$$v_0 \approx \frac{Z(t_2) - Z(t_1)}{t_2 - t_1} = 11.3 \text{ m/s}.$$  

We find the appropriate $t$ ranges by assuming that standard model remains positive. Thus, we solve $v_0 t - \frac{1}{2}(9.81t^2) = 0$, getting

$$t = 0.000 \text{ s} \quad \text{and} \quad t = \text{standard landing time} = 2.31 \text{ s}.$$
We plot and compare our model to the standard parabolic model (Figure 21).

Figure 21. Height model (solid line) from sixth-degree least-squares fit vs. standard parabolic model (dashed line) estimating $v_0$ from $t_1$ and $t_2$.

The greater height of the parabolic model reflects the difference in thrust in the two models.

We look at the difference in the air time, which will give us a sense of how the wind may have held the rocket in the air long than might be expected, or driven it into the ground earlier than might be expected.

$$\text{air time difference} = |t_{\text{land}} - \text{standard landing time}| = 0.0564 \text{ s}.$$

The rocket was accelerating from $t_1$ to $t_2$, so the initial velocity $v$ used above may not be the best parameter for a good comparison. If the initial value shifts to $(t_i, z_i)$, the model becomes $S(t) = v_0(t - t_i) - \frac{1}{2}g(t - t_i)^2 + z_i$. We estimate the parameter $v$ using times $t_2$ and $t_3$, since the thrusting stage was about over by then. In this case, we get the plot of our model vs. the standard model as shown in Figure 22.

Comparing these two models, we get an air time difference of 0.209 s in this case. This model gives a better fit for the first part of the flight; and since the rocket was in fact blown by the wind, this is probably a more useful comparison than the parabolic model derived using $t_2$ to $t_3$.

6. Some Comments on the Modeling

Hindsight is 20/20. In the process of running this project with our classes, repeating it to get the data presented here, and then writing this Module, we identified problem areas, thought of ideas for improving the project, and discovered new resources. We plan to use some of these items in the future, and offer them here for consideration. We hope that anyone who tries this Module, and especially anyone who experiments with any of the modifications below, will contact us to share their experiences.
Figure 22. Height model (solid line) from sixth-degree least-squares fit vs. standard parabolic model (dashed line) estimating $v_0$ from $t_2$ and $t_3$.

6.1 Measurements

The task of taking accurate measurements in this experiment is difficult because of uneven turf and uncertain elevations. These inaccuracies can have a considerable effect on the model. Furthermore, the method used to estimate the launching and landing positions of the rocket (ground measurements) was different from the method used to estimate the in-flight positions (projection against the building). This must be done carefully to avoid misleading results. More sophisticated measuring tools, if available, could help here. For example, the use of laser telemetric devices (e.g., electronic theodolites [Mohave Instrument 2003]) would improve the measurements of distance.

6.2 Sensitivity

It is instructive to see how robust our numerical results are by introducing a measurement uncertainty in our perceived positions of the rocket during its flight path. First, we look at how uncertainties in the position of the water-rocket along its path can change the maximum height achieved by the water rocket. In Figure 23 (where the parabolic curves are generated from the first two times), we show the influence of a 1% uncertainty (Figure 23a) and a 10% uncertainty (Figure 23b) artificially imposed on the data. A 10% uncertainty, which can result from making measurements to two significant digits, shows tremendous variations in maximum height and, by extension, maximum horizontal distance covered. In contrast, a 1% uncertainty, which can result from making measurements to three significant digits, shows only small variations.

Next, we look at the effect of measurement uncertainties on curvature, which provided above a clear signal for a gust of wind occurring at approximately 1.9 s after the launch. While a 1% uncertainty in the data measurements shows virtually no changes in the curvature graph, a 10% uncertainty
Figure 23. Influence of measurement error on flight path (solid line) and on standard model (dashed line), for 1% error and 10% error.

(as shown in Figure 24) clearly shows variations in the maximum curvature achieved during the flight path of the water rocket. We also note, however, that the wind-gust feature at 1.9 s is fairly robust.

Figure 24. Effect of 10% measurement error on curvature.

6.3 Flight Stages

Because rocket propulsion allows the water rocket to leave the ground with zero initial velocity, the forces acting on the rocket during the thrusting stage are different from those during the coasting and recovery stages. Since the polynomial fit presented here is a global fit (i.e., it assumes that the physics of the initial thrusting stage is the same as the later two stages), it mistakenly assigns a nonzero initial velocity. By taking more data points, it should be possible to develop two separate approximating functions, one for the thrusting stage (with zero initial velocity), and one for the coasting and recovery stages (with
nonzero initial velocity at the beginning of the coasting stage). This would give a piecewise function as a model of the rocket flight, although presumably a more accurate one. The disadvantage of possibly discontinuous derivatives might be addressed with curve smoothing techniques.

6.4 Weathercocking

Weathercocking is the cumulative effect of the airflow over the nose of the rocket and the wind. This effect can cause the rocket to move into the wind. There is a good description of weathercocking at Folger [2001]. Because the wind was gusting during our launch, we were not able to get a sense of what effect this might have had on our rocket. A water rocket launched during a gentle, steady wind might demonstrate some weathercocking. It would be interesting to try to determine the extent of this effect. It might also be useful to record the wind direction and speed during the flight. The technical difficulties are finding a practical means for doing so and correlating the wind data with the flight data.

6.5 Three-Dimensional Coordinates

We have always done this project using a single video camera. This has forced us to make the simplifying assumption that the rocket remains in the vertical plane through the ground path. Using two widely spaced video cameras to record the launch would allow the construction of a nonplanar space curve. It would require careful synchronization of the recordings. More critically, significantly more work would need to be done to estimate the actual rocket positions from its two different apparent positions against the building from the video stills of the two different cameras. This would shift the bulk of the computation in the project to three-dimensional coordinate geometry and away from the calculus of three variables that we wanted to emphasize. However, it would presumably give a more interesting space curve as a model, and would make considering other characteristics, such as the pitch, yaw, and roll, of the rocket more meaningful. The debate continues, and we would be very glad to hear from anyone who tries this “binocular” version!

6.6 Measuring Maximum Height with a Simple Sextant

Although the videotape is not much use in estimating the height of the rocket once it has soared above the building, a simple sextant can be used to determine the height at the apex. This device measures the angle of elevation from eye level to the object sighted through the straw. An observer at a known position relative to the rocket launch site measures the angle of elevation to the rocket when it reaches the apex. The projection of the rocket’s position onto
the ground must be determined, and the distance from there to the observer computed. That distance and the angle of elevation can then be used to estimate the height. We suggest having at least two people make observations and then average their estimates of the angular distance. If the observers shout “Max!” when they make their observations, the sounds of their voices will be recorded on the videotape, which may give a sense of how close to the actual apex the observations were made.

It would be fun to reserve these data initially and construct a model without them. Then they could be used to see how accurately the model predicts the maximum height. If the model does a poor job, these data could be incorporated into the model, and then analysis done on the improved model.

Figure 25 shows a diagram of a simple sextant; the diagram and instructions below (slightly modified) come from Folger [2001].

![Diagram for a simple sextant.](image)

**Materials**
- a large-diameter soda straw
- 20 cm length of string
- a protractor
- a weight (an eraser or large washer)
- tape

**Construction**
Tape the straw across the top of the protractor as shown in the figure. The straw will act as a sighting tube. Secure the string to the protractor by slipping
it under the straw and around. Tie the string to itself and tape it to the back of the protractor. Tie the eraser or washer at the opposite end of the string, so that it can act as a weight.

**Directions for Use**

Look through the straw, focusing on the rocket as it is being launched. Let the weight hang freely but try not to let it swing. Move the device up smoothly as the rocket ascends. At the instant the rocket turns around, hold the string with your finger exactly where it is on the protractor. Record this angle.

### 7. Appendices

#### 7.1 Additional Video Stills

We strongly recommend launching and videotaping your own rocket. It is a lot of fun, and much more satisfying to analyze your own data. However, in case that is simply not feasible, we can provide some additional video stills from the same launch as the example in this paper. Please email us for these. There is a scale at the top of the blueprint in Figure 3 in Section 3.1 for estimating the apparent position of the rocket against the building.

#### 7.2 Maple 8 Code

Also available from us via email request is the Maple 8 code that we used. Here we briefly outline our approach to the code.

We first provide generic Maple procedures, which may be used with any rocket flight data set, followed by commands that generate the specific results presented in this Module.

To format the labeled figures in this Module, we used Maple plot options to set parameters such as `tickmarks`, `labelfont`, `axesfont`, `title`, `labeldirections`, and `orientation`; however, for improved readability, we have omitted most of these labeling options from the code.

In Maple, procedures must be executed within a single execution group. The output from a procedure is usually in the form of a list, `ReturnVal`, found at the end of the procedure. Recall that in Maple, if `L` is a list, then `L[i]` returns the `i`th entry in the list. Also the command `nops`, for the number of operands of an expression, when applied to a list, returns the number of items in `L`. Thus, if `L := [3, 4, [5, 6]]`, then `L[2] = 4`, `L[3] = [5, 6]`, and `nops(L) = 3`. Also, recall that the `unapply` command may be used in Maple to convert an expression into an operator.

Maple carries out all computations to 10 decimal places. However, our data are accurate to only three significant digits. Thus, in our output, we use the `evalf` command to restrict our displayed results to three significant digits.
7.3 Curve Fitting

Details on creating interpolated and least-squares fit models can be found in Sections 6.4 and 9.3 of Anton [1994].

References and Resources


This Website also offers a “Beginner’s Guide to Aerodynamics” and a “Beginner’s Guide to Model Rockets.” The model rocket information includes some of the basic mathematics and physics concepts that affect model rocket design and flight. The site is intended to offer K-12 teachers, students, and others with basic background information on aerodynamics and propulsion.


Estes Industries for over 40 years has been “the world’s leader in the manufacturing of safe, reliable, and high quality rocket products” and offers a variety of “new and innovative educational products.” This site is a comprehensive source of information about how model rocketry can be integrated into the classroom; it offers links to a number of different publications (in PDF format) on model rocketry, ranging from teachers’ guides to elementary mathematics information on rocket flights.


“Dave’s Cool Toys” is a Website at which water-powered rockets can be purchased inexpensively. The link sequence Our Toys —> Outside leads to ordering information about water-powered rockets as well as higher-flying “meteor rockets” (powered by a baking soda and vinegar solution) and even higher-flying “air burst rocket systems” (powered by compressed air).


This paper looks at the effects of air resistance and thrust on the one-dimensional path of a water-propelled rocket launched vertically.

The Future Astronauts of America Foundation is “a multi-faceted program designed to encourage young students and adults to become involved with the exciting world of space science and its applications.” This site provides information about model rocket components. It also offers useful descriptions of the flight sequence of a model rocket, including its thrusting, coasting, and recovery stages.


This site contains information about commercially available electronic theodolites.


This Website is “an exploration environment for concepts in physics which employs concept maps and other linking strategies to facilitate smooth navigation.” It contains useful information on rocket propulsion.


This paper details a general model for a water rocket flight and offers advanced analysis of the trajectory. The paper is suitable for students with applied mathematics or engineering interests, at the advanced undergraduate level or beginning graduate level (for example, in a fluid dynamics course). It also offers a good application for an advanced graduate-level course in computational methods.

About the Authors

George Ashline received his B.S. from St. Lawrence University, his M.S. from the University of Notre Dame, and his Ph.D. from the University of Notre Dame in 1994 in value distribution theory. He has taught at St. Michael’s College since 1995. He is a participant in Project NExT, a program created for new or recent Ph.D.s in the mathematical sciences who are interested in improving the teaching and learning of undergraduate mathematics.

Alain Brizard …

Joanna Ellis-Monaghan received her B.A. from Bennington College, her M.S. from the University of Vermont, and her Ph.D. from the University of North Carolina at Chapel Hill in 1995 in algebraic combinatorics. She has taught at Bennington College, at the University of Vermont, and since 1992 at St. Michael’s College. She is a proponent of active learning and has developed materials, projects, and activities to augment a variety of courses.