INTERMODULAR DESCRIPTION SHEET: UMAP Unit 777

TITLE: Population Models in Biology and Demography

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MATHEMATICAL FIELD: Calculus

APPLICATION FIELD: Demography, biology

TARGET AUDIENCE: Students in first-semester calculus.

ABSTRACT: This module presents applications from microbiology and demography that give a physical context for critical concepts in a first-semester calculus course. Elementary models of population growth are developed using data collected from biological experiments conducted during the course. Then, proceeding from "microcosm to macrocosm," the tools of calculus, demographic software, and the Maple computer algebra system (CAS) are used to address questions of human population projections. Included are discussion materials, sample models using exponential and logistic curves, projects, complete biology labs for generating raw data, exercises for developing facility with Maple, and resource lists (including software and Internet sites) for studying demographic questions.

PREREQUISITES: None.

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Microcosm to Macrocosm: Population Models in Biology and Demography

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1. Introduction

The main objective of this Module is to use biology and demography to provide interesting, motivating contexts for the material in a first-semester calculus course. The activities demonstrate how a mathematical model can address complex questions and yield answers simply not discernible from the raw data alone. This connection strongly motivates the use of modeling and the importance of mathematics in studying biological and sociological issues (and, by extension, issues in other fields).

The Module begins by analyzing biological growth and concludes with a major research project that investigates how laboratory results can be extended to consider the question of human population change. Proceeding from microbiology to demography demonstrates how laboratory experiments can serve as a microcosm for larger questions. The activities give experience in the process of questioning and validating mathematical techniques. They also give exposure to some of the inherent difficulties of modeling population changes.

The extremes resulting from choosing an inappropriate model for human population change can be seen in the following rather lurid scene from the 1973 dystopian science fiction film *Soylent Green* [Fleisher 1973]: The year is 2022 A.D. Forty million people crowd New York City. An air-locked plastic bubble preserves the small handful of sickly trees remaining. Bodies pack stairwells at night. Only the very wealthy can afford fresh vegetables, strawberry jam, or even hot running water. Teeming masses of desperate people mill around derelict cars on filthy streets, wearing surgical masks against the thick yellow smog. They fight for crumbs of government-supplied protein crackers, and bulldozers control daily food riots, scooping crying, starving people into bloody piles. Sanitation trucks arrive for routine collection of the dead.

Is this be a realistic vision of the future or not? Who would predict such population growth and on what basis? In the prologue to the book that was the basis for *Soylent Green*, author Harry Harrison writes that

> [W]ithin fifteen years, at the present rate of growth, the United States will be consuming over 83 per cent of the annual output of the earth's materials. By the end of the century, should our population continue to increase at the same rate, this country will need more than 100 per cent of the planet's resources to maintain our current living standards. This is a mathematical impossibility–aside from the fact that there will be about seven billion people on this earth at that time and–perhaps–they would like to have some of the raw materials too. [Harrison 1966, ???]

need page

Now, more than 30 years later, Harrison's prediction has clearly been disproved. His statement shows that he based his vision of the future on an exponential model of human population growth. Throughout history, many people have chosen, either deliberately or out of ignorance of other models, to use a particular population model specifically to make a case for a particular point of view. For example, Paul Ehrlich used an exponential model to predict that in

900 years world's population will be 60,000,000,000,000,000, or 100 people per square yard of the earth's surface, land, and sea [1968, 18]. He then used this dire prediction to justify such draconian measures as involuntary sterilization of the general population through contraceptives in the water supply, with the antidote "carefully rationed by the government to produce the desired population size" [1968, 135], although he admitted such measures would be difficult to enforce.

In fact, not only are populations in many parts of the world not growing exponentially, they aren't even growing—they are declining. The replacement growth rate (the rate needed for an unchanging population) is about 2.2 children born to each woman [Cohen 1995, 288]. However, in 1997 the rate in both Spain and Italy was only 1.2 children per woman. Europe as a whole is well below replacement level, with an average total fertility rate of 1.4. Even the United States is at only 2.0 children per woman [Population Reference Bureau 1999]. Many other countries have fertility levels below the replacement level, and it is expected that this trend will also eventually manifest itself in many of the remaining countries [Tapinos and Piotrow 1980, 168–169].

So, although exponential models can be very helpful if used appropriately, (and these models will be developed in this Module), as Cohen points out, they have repeatedly failed as good forecasters of long-term population growth:

> Because of its great simplicity, the exponential model is remarkably useful for very short-term predictions: the growth rate of a large population during the next one to five years usually resembles the growth rate of that population over the past one to five years. Because of its great simplicity, the exponential model is not very useful for long term predictions, beyond a decade or two. Surprisingly, in spite of the abundant data to the contrary, many people believe that the human population grows exponentially. It probably never has and probably never will. [Cohen 1995, 84]

Why is population change such an urgent concern? Who wants to know? And why is it so hard to predict? Carl Haub addresses the first two questions:

> Interest in [population] projections involves much more than a simple curiosity about what may lie ahead. Having some sense of the number of people expected, their age distribution, and where they will be living provides city planners and local governments, for instance, sufficient "lead time" to prepare for coming needs in terms of schools and traffic lights, or reservoirs and pipes to deliver water supplies. Businesses have a vital interest in the coming demand for their products and services.
> [Haub 1987, 3]

World population projections have been a cause of considerable concern, particularly because of the population "explosion" of the post-World War II years. The probable consequences of rapidly expanding human numbers have been the subject of lively debate. Recently, that debate has been joined in controversy by the issue of population decline resulting from the very low birth rates in some countries. Books on these concerns range from Ehrlich's *The Population Bomb* [1968] to Ben Wattenberg's *The Birth Dearth* [1987].

Despite the urgency of the questions, answers to human population concerns are hard to determine. Complicating the prediction process is the fact that the forecasters may themselves be biased, influencing their choice of models and their basic assumptions. As Julian Simon notes,

> [S]everal groups have had a parochial self-interest in promoting these doom-saying ideas [of explosive population growth leading to scarce resources]. . . . [T]hese groups include (a) the media, for whom impending scarcities make dramatic news; (b) the scientific community, for whom fears about impending scarcities lead to support for research that ostensibly will ease such scarcities; and (c) those political groups that work toward more government intervention in the economy; supposedly worsening scarcities provide an argument in favor of such intervention.
> [Simon 1990, 3–4]

Demographers honestly admit that they cannot predict the future: "[M]ost professional demographers no longer believe they can predict precisely the future growth rate, size, composition and spatial distribution of populations" [Cohen 1995, 109–110]. Furthermore, any mathematical model hoping to predict future conditions can only assume that existing trends will continue unchanged or will change in a predictable way. This is a very tenuous assumption, especially regarding such factors as fertility and mortality rates.

People themselves create the major difficulty in applying these models to the human condition. Unlike bacteria in a laboratory which simply reproduce until they have exhausted all available nutrients, humans have highly complex value systems which effect their reproductive trends and use of resources.

According to Cohen,

> Humans seem to resolve conflicts of values by personal and social processes that are poorly understood and virtually unpredictable at present. How such conflicts are resolved can materially affect human carrying capacity, and so there is a large element of choice and uncertainty in human carrying capacity. . . . Not all of those choices are free choices. Natural constraints restrict the possible options . . . . [Cohen 1995, 296]

Because of this,

> Estimating how many people the Earth can support requires more than demographic arithmetic. . . . [I]t involves both natural constraints that humans cannot change and do not fully understand, and human choices that are yet to be made by this and by future generations. Therefore the question "how many people can the Earth support?" has no single numerical answer, now or ever. Because the Earth's human carrying capacity is constrained by facts of nature, human choices about the Earth's human

> carrying capacity are not entirely free, and many have consequences that are not entirely predictable. Because of the important roles of human choices, natural constraints and uncertainty, estimates of human carrying capacity cannot aspire to be more than conditional and probable estimates . . . . [Cohen 1995, 261–262]

But decisions must be made, by the business community, by governments, by policy makers, and by individuals. These decisions depend on estimates of future conditions. So how can population trends be predicted, to allocate resources for example? And how is the choice of model justified and its effectiveness evaluated? To address these questions, it is first necessary to know what models are available and why they might be applied to human population changes. It is also necessary to acquire the mathematics to evaluate and manipulate these models. The first-semester calculus course develops the necessary mathematical tools to understand exponential and logistic curves. These curves can be used as models to analyze the growth patterns demonstrated in the laboratory component of this module. The collection of data from laboratory cultures of various organisms shows that both functions are quite effective models of population growth. In fact, both can even give quite accurate predictions of future growth.

So, it seems reasonable to apply these models to the human populations of various countries, and the culminating project of this Module provides the opportunity to do just that. Human population data can be collected from a variety of sources, including Internet sites giving the most up-to-date information currently available. Short-term predictions can be compared to known results and the new data used to refine the models. Finally, the models can be used to predict future populations, both for individual countries and for the entire world population. Internet resources are available to see how these predictions compare to the most recent estimates of professional demographers. For most areas of the world, these models prove especially good for backward analysis and passable for very short-term predictions of future change. However, they usually have limitations as long range forecasters of human populations. References to other models that require more advanced mathematical techniques are given in Sections 5.2–5.3 for those who are interested in pursuing these questions.

2. Modeling Examples

We develop models for a bacteria colony in a controlled setting and for the population of Italy.

2.1 Raw Data

2.1.1 Bacteria Colony

We present data for a bacteria culture in the spring of 1995. The bacteria were grown in a petri dish, and every few days the diameter of the bacteria culture was measured. **Table 1** gives the recorded data points, where t_i is the time (in hours, converted to a decimal, after 12:45 PM on 4/11/95), d_i is the diameter (in millimeters), and $a_i = \pi d_i^2/4$ is the area covered by the colony. **Figure 1** shows the area as a function of time.

Table 1.
Data for a bacteria culture.

Index	Time (hours)	Diameter (mm)	Area (mm^2)
0	0	2.5	4.91
1	23.17	5	19.63
2	49.92	5	19.63
3	98.5	6	28.27
4	148.85	6.6	34.21
5	167.5	6.75	35.78
6	192.17	6.52	33.39

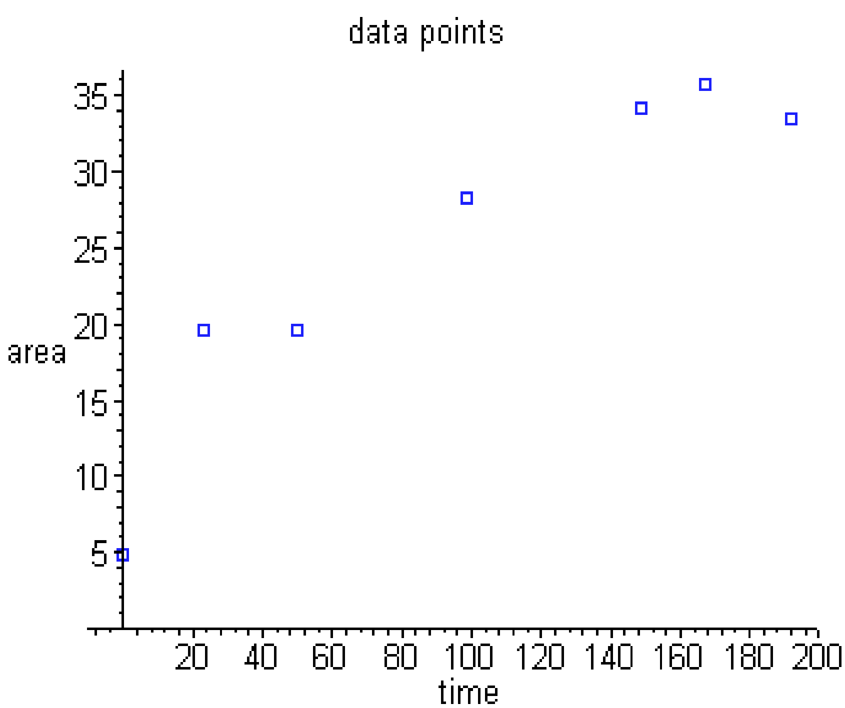

Figure 1. Area of bacteria culture (mm^2) vs. time (hours).

2.1.2 Population of Italy

The census data from Italy in this second example interestingly shows a similar growth pattern. The similarity suggests that a model developed for bacteria growth might be applicable to human population. **Table 2** and **Figure 2** give the population of Italy for several years prior to 1950, taken from various editions of *World Almanac and Book of Facts*.

Table 1.
The population of Italy.

Date	Population
1915	35,240,000
1921	37,270,493
1928	41,168,000
1931	42,118,835
1936	42,527,561
1940	45,330,441
1943	45,801,000
1946	45,646,000
1948	45,706,000

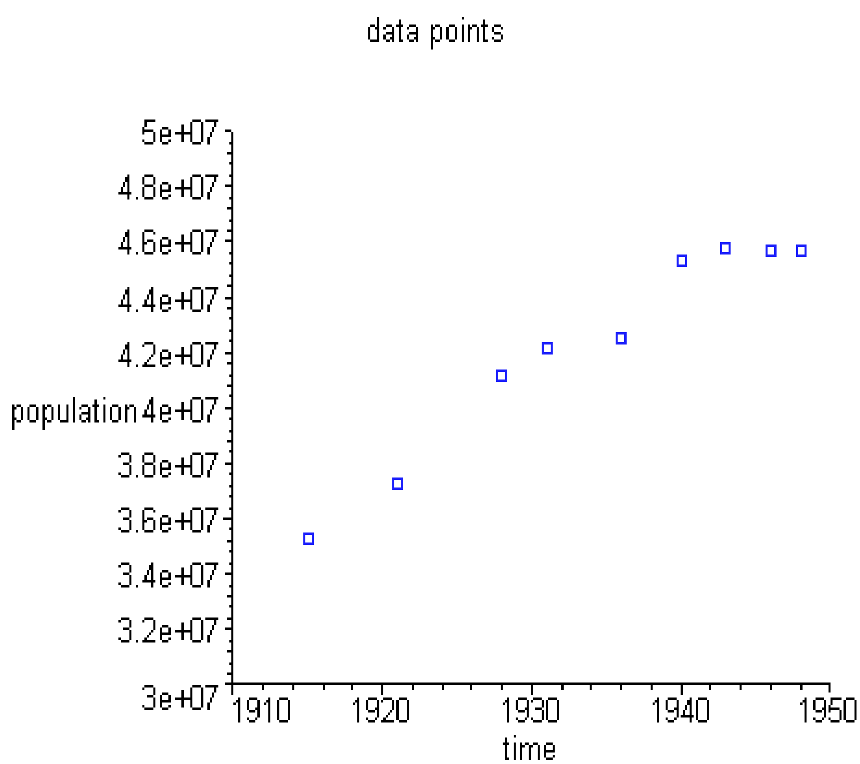

Figure 2. The population of Italy.

2.2 Exponential and Logistic Functions

To model data such as that in the examples above, it is necessary to be familiar with functions whose graphs are similar to the shapes displayed by the data. The following functions are useful for modeling certain kinds of growth, including the examples of bacteria growth and the population of Italy.

The functions and graphics below, and those in the following section of modeling examples, are given using Maple, a computer algebra system. Section 4, **Developing Maple Facility**, contains more information and examples of Maple code.

Exponential growth: The exponential function $y(t) = Ce^{kt}$, where C is the initial amount and k is the growth constant, models unrestricted growth with the rate of growth proportional to the amount present. Thus, the function satisfies the differential equation $y' = ky$. For example, **Figure 3** shows a plot of $y(t)$, with $C = 5.05$ and $k = .023$, generated from the following Maple code:

```
> y:=t->C*exp (k*t):
> C:=5.05:  k:=0.023:
> plot(y(t), t=-50..100);
```

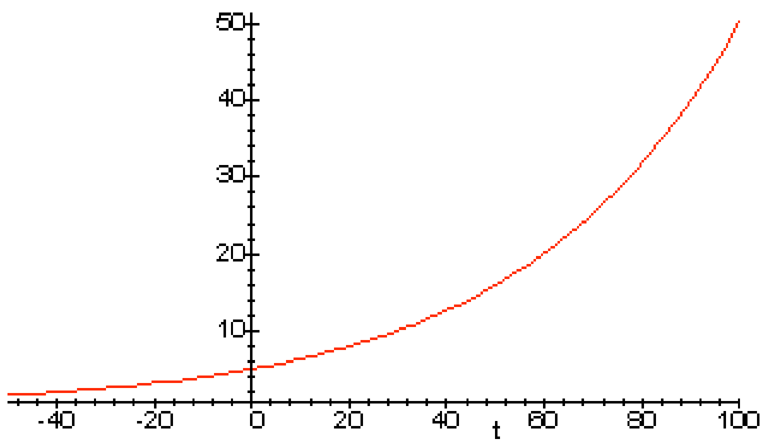

Figure 3. Exponential growth.

Limited growth: The limited growth curve models bounded growth, where the rate of growth is proportional to how close the amount is to the carrying capacity of the system. Such a curve looks like $y(t) = M(1 - e^{-kt})$, where M is the carrying capacity and k is the growth constant; it satisfies the differential equation $y' = k(M - y)$. **Figure 4** shows a plot of $y(t)$, with $M = 15$ and $k = 1.23$, generated from the following Maple code:

```
> y:=t->M*(1-exp(-k*t)):
> M:=15:  k:=1.23:
> plot(y(t), t=-1..4, y=-10..18);
```

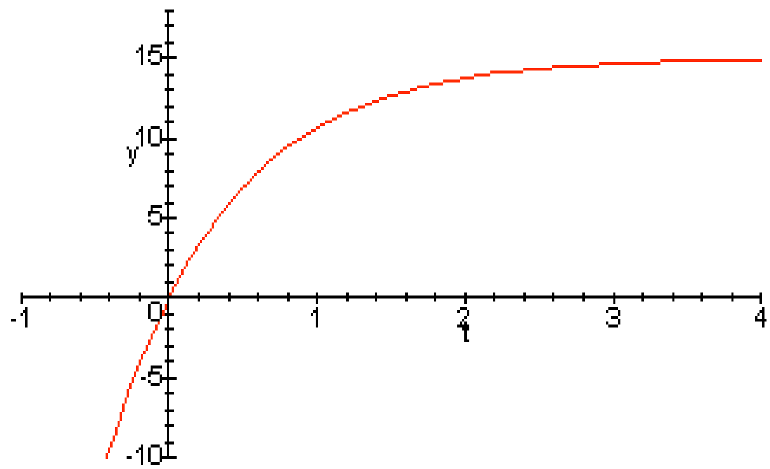

Figure 4. Limited growth.

Logistic growth: The logistic curve models growth where the rate of growth is proportional to both the amount present and the difference between the carrying capacity and the amount present. A logistic function looks like $y(t) = M/(1+Be^{-Mkt})$, where M is the carrying capacity, B is a number that depends on the carrying capacity and the initial amount, and k is the growth constant; it satisfies the differential equation $y' = ky(M-y)$. **Figure 5** shows a plot of $y(t)$, with $M = 25$, $B = 22$, and $k = 0.23$.

```
> y:=t->M/(1+B*exp(-M*k*t)):
> M:=25:  B:=22:  k:=0.23:
> plot(y(t), t=-1..3);
```

2.3 Modeling Bacteria Growth

We use a logistic curve to model the growth of the bacteria culture. We enter the raw data into Maple:

```
> s:=evalf(12+45/60):  t[0]:=0:  d[0]:=2.5:  t[1]:=23.0+10/60:
d[1]:=5:  t[2]:=48.0+2-5/60:  d[2]:=5:  t[3]:=4*24+2.5:  d[3]:=6:
t[4]:=6*24+4.0+51/60:  d[4]:=6.6:  t[5]:=7*24-.5:  d[5]:=6.75:
t[6]:=8*24.0+10/60:  d[6]:=6.52:
```

Then we compute the areas a_i using a loop

```
> for i from 0 to 6 do
>      a[i]:=Pi*.25*d[i]^ 2
> od:
```

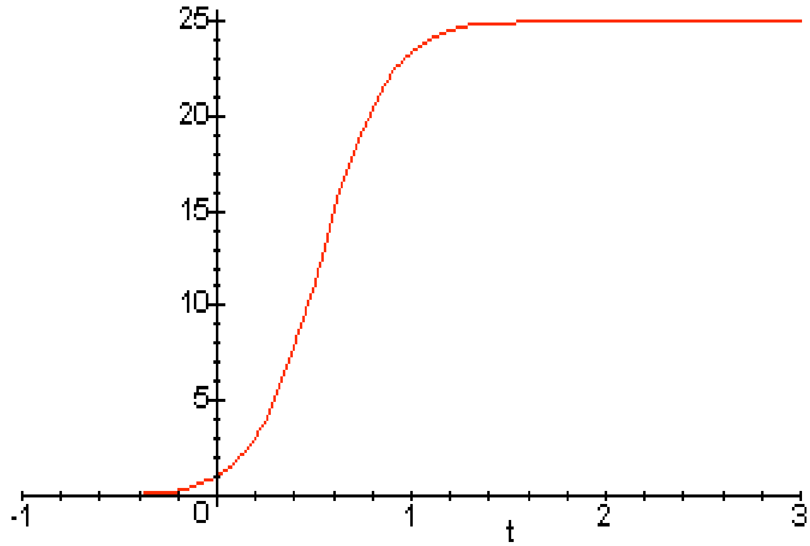

Figure 5. Logistic growth.

Pairs of data points, together with an estimate for the carrying capacity M, are used to determine a logistic function $L(t)$ that models the growth of the bacteria. The general form of $L(t)$ is:

```
> L:=t-> M/(1+B*exp(-M*k*t));
```

$$L(t) = \frac{M}{1 + Be^{-Mkt}}$$

A good way to get a preliminary estimate for M is to plot the data points and do a freehand sketch of an "S" shape through them. Descriptions of more-refined techniques for determining the values of the parameters can be found in the articles on curve-fitting in Section 5.3.

We get the plot of **Figure 6** by inputting:

```
> with(plots):
> A:=plot([ [t[0],a[0]], [t[1],a[1]], [t[2],a[2]], [t[3],a[3]],
[t[4],a[4]], [t[5],a[5]], [t[6],a[6]] ], x=-10..200, y=0..a[5]+1,
style=point, title=‘data points‘, symbol=box, labels=[time,area]):
> display(A);
```

The second data point looks a little too high, and the last one a little too low, probably due to inaccuracy in measurement. A good guess for the carrying capacity M seems to be 34.5, so now $L(t)$ looks like:

```
> M:=34.5:  L(t);
```

$$L(t) = \frac{34.5}{1 + Be^{-34.5kt}}$$

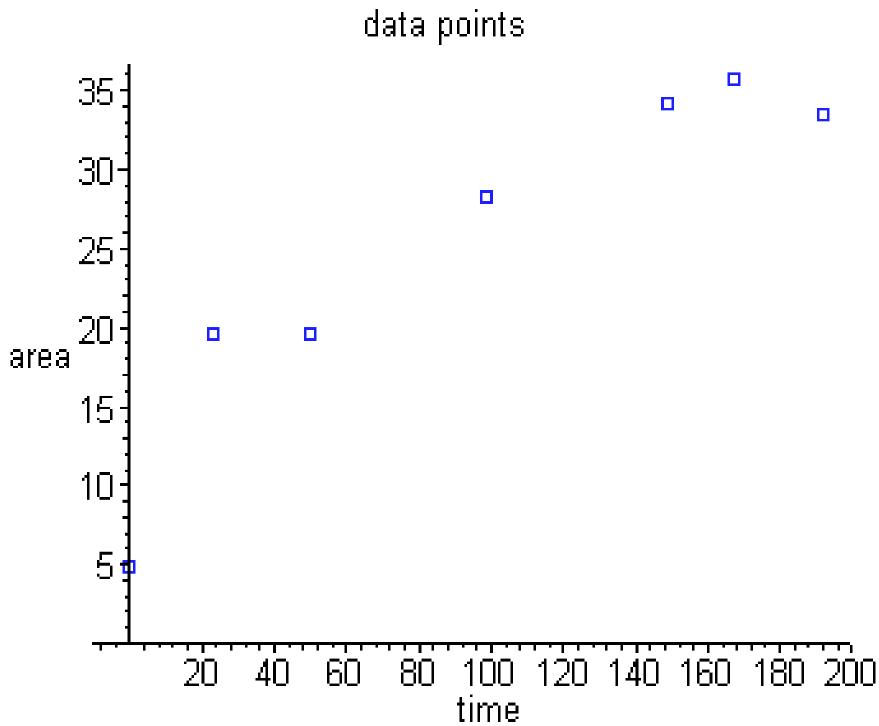

Figure 6. Logistic fit to bacteria growth.

Now, using t_0 and a_0 to solve for B gives:

```
> solve(a[0]=L(t[0]),B): B:=%;
```

$$B = 6.028282287$$

Thus, $L(t)$ becomes:

```
> L(t);
```

$$L(t) = \frac{34.5}{1 + 6.028282287e^{-34.5kt}}$$

Next, t_2 and a_2 are used to find k; the data point (t_1, a_1) looks as though it might be from a bad measurement, so (t_2, a_2) will probably yield a better model:

```
> solve(a[2]=L(t[2]),k);
```

$$0.001204767896$$

```
> k:=%;
```

With this, $L(t)$ becomes:

```
> L(t);
```

$$L(t) = \frac{34.5}{1 + 6.028282287e^{-0.0415649241t}}$$

Now the model can be compared to the full data set. Here are the values of the function $L(t)$ and the actual areas at the times t_0 through t_6:

```
> seq(evalf(L(t[i])), i=0..6);
```
4.908738521, 10.44986633, 19.63495408, 31.34950928, 34.07758738, 34.30413399, 34.42948532
```
> seq(evalf(a[i]), i=0..6);
```
4.908738522, 19.63495409, 19.63495409, 28.27433389, 34.21194400, 35.78470382, 33.38759009

All but the second are fairly close. The following command generates **Figure 7**, which shows how the actual data points fit on the curve:

```
> display(A,BC);
```

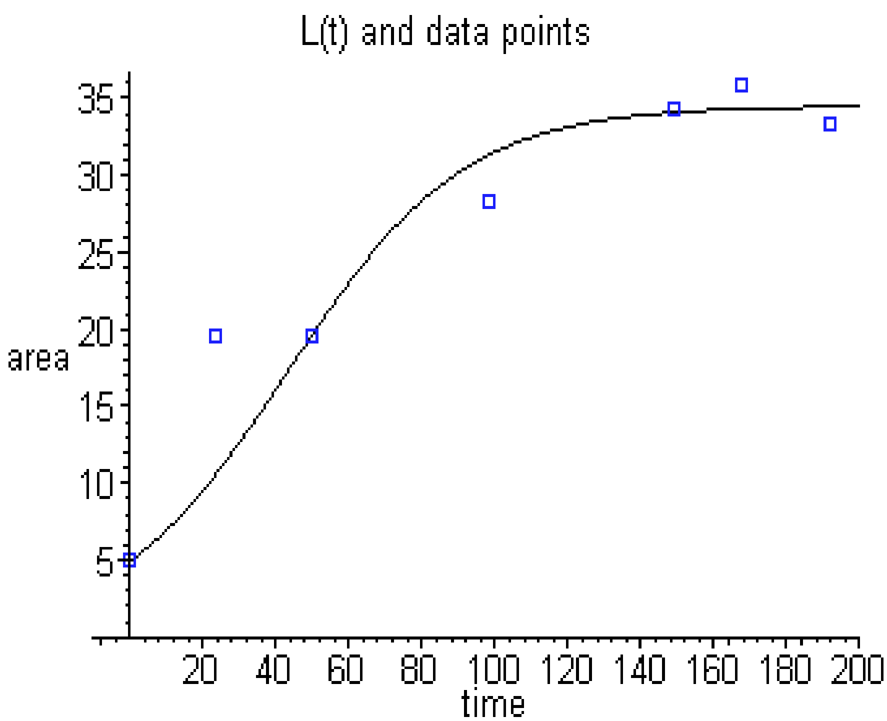

Figure 7. The data points and the model.

The most important aspect of having developed a model is that it is now possible to answer some questions that could not have been determined just using the observed data.

- *What was the area covered by the colony at 10:00 A.M. on 4/16?*

  First, it is necessary to determine the value of t at that time by counting 5 days less 2:45 hours from the start of the experiment at 12:45 on 4/11. This gives:

```
> t[S]:=5*24-2.75;
```
$$t_s = 117.25$$

Now, evaluating $L(t)$ at that time gives the area covered:

```
> L(t[S]):
```

$$32.97968560$$

- *At what rate is the area increasing at that time?*

The derivative of $L(t)$ is needed to answer this. Fortunately, $L(t)$ satisfies the differential equation $y' = ky(M - y)$, which gives the following formula for $L(t)$:

```
> Lprime:=t-> k*(L(t))*(M-L(t));
```

$$Lprime := t \to kL(t)(M - L(t))$$

Evaluate at t_s to find the rate:

```
> Lprime(t[S]);
```

$$.06040644899$$

- *What is the carrying capacity of the system?*

The carrying capacity is just M.

```
> M;
```

$$34.5$$

- *When does the rate slow down? In other words, when does the population stop growing at an increasing rate and start growing at a decreasing rate? (This is just asking where the inflection point is).*

A good estimate for this can be found by inspecting the graph: about 43 hours after the start of the experiment. Alternatively, the second derivative of $L(t)$ can be set equal to zero, and solving for t gives the t-coordinate of the inflection point:

right syntax?

```
> diff(L(t),t$2):
> solve(%=0,t);
```

$$43.22107658$$

- *What is the average area covered by the colony during the course of the experiment?*

First, integrating $L(t)$ from t_0 to t_6 gives the total area:

```
> int(L(t), t=t[0]..t[6]);
```

$$5012.92708$$

Then, dividing this by $t_6 - t_0 = t_6$ yields the average area (in mm^2) covered by the colony during the course of the experiment:

```
> int(L(t),t=t[0]..t[6])/t[6];
```

$$26.08635080$$

2.4 Modeling the Population of Italy

We use a logistic curve to model the population of Italy from 1915 to 1948, then consider how well the model predicts the population later.

We input the population of Italy for the nine years shown in the chart in Section 2.1, as well as for six later dates:

```
> restart;
> t[0]:=1915:  p[0]:=35240000:  t[1]:=1921:  p[1]:=37270493:
t[2]:=1928:  p[2]:=41168000:  t[3]:=1931:  p[3]:=42118835:
t[4]:=1936:  p[4]:=42527561:  t[5]:=1940:  p[5]:=45330441:
t[6]:=1943:  p[6]:=45801000:  t[7]:=1946:  p[7]:=45646000:
t[8]:=1948:  p[8]:=45706000:  t[9]:=1960:  p[9]:=50763000:
t[10]:=1965:  p[10]:=52736000:  t[11]:=1970:  p[11]:=53670000:
t[12]:=1975:  p[12]:=55810000:  t[13]:=1980:  p[13]:=57040000:
t[14]:=1990:  p[14]:=57657000:
```

We use two data points and an estimate for M to create a logistic function $L(t)$ to model the population of Italy with the pre-1950 population data. A logistic curve with 1915 as its initial date has the following form, for some constants M, B, k:

```
> L:=t-> M/(1+B*exp(-M*k*(t-1915)));
```

$$L(t) = \frac{M}{1 + Be^{-Mk(t-1915)}}$$

As before, M can be estimated from a plot (**Figure 8**):

```
> with(plots):
>A:=plot([[t[0],p[0]],[t[1],p[1]],[t[2],p[2]],[t[3],p[3]],
[t[4],p[4]],[t[5],p[5]], [t[6],p[6]],[t[7],p[7]], [t[8],p[8]]],
x=1900..2000, y=0..60000000, style=point, title=`data points`,
symbol=box, labels=[time, population]):  display(A);
```

A good guess for M from these points seems to be 50 million, so that $L(t)$ becomes:

```
> M:=50000000:  L(t);
```

$$L(t) = \frac{50000000}{1 + Be^{-50000000k(t-1915)}}$$

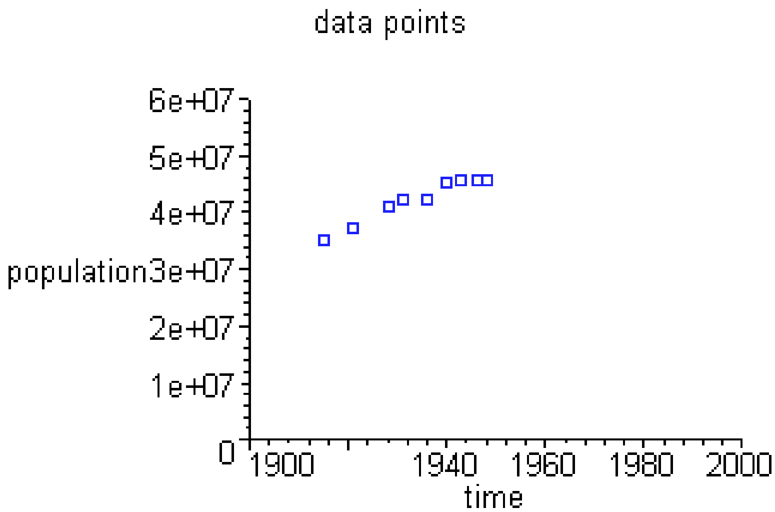

Figure 8. Estimating M from a plot.

Next, using t_0 and p_0 to solve for B gives:

```
> solve(p[0]=L(t[0]),B):
> B:=evalf(%);
```

$$B := .4188422247$$

Now L(t) looks like:

```
> L(t);
```

$$L(t) = \frac{50000000}{1 + .4188422247 e^{-50000000 k (t-1915)}}$$

In turn, t_5 and p_5 are used to find k. This data point is used since it is part of a group of points toward the end of the modeling period:

```
> solve(p[5]=L(t[5]),k);
```

$$.1122122542\ 10^{-8}$$

> k:=%:

Thus, after simplifying, $L(t)$ looks like:

```
> L(t);
```

$$L(t) = \frac{50000000}{1 + .4188422247 e^{-.05610612710 t + 107.4432e34)}}$$

The values of the function $L(t)$ at the times t_0 through t_8 and the actual populations prior to 1950 are:

```
> seq(evalf(L(t[i])), i=0..8);
```

$.3523999999\ 10^8$, $.3848747902\ 10^8$, $.4159840763\ 10^8$, $.4271018355\ 10^8$,

.4428974587 10^8, .4533044101 10^8, .4599589675 10^8, .4657368685 10^8,
.4691493452 10^8

```
> seq(evalf(p[i]), i=0..8);
```

.35240000 10^8, .37270493 10^8, .41168000 10^8, .42118835 10^8,
.42527561 10^8, .45330441 10^8, .45801000 10^8, .45646000 10^8,
.45706000 10^8

Thus, the modeling function seems fairly accurate up to 1950. **Figure 9** shows how the data points fit this curve:

```
> BC:=plot(L(t),t=1900..2000,y=0..60000000,
labels=[time,population], color=blue, title=‘L(t)‘):
display(A,BC);
```

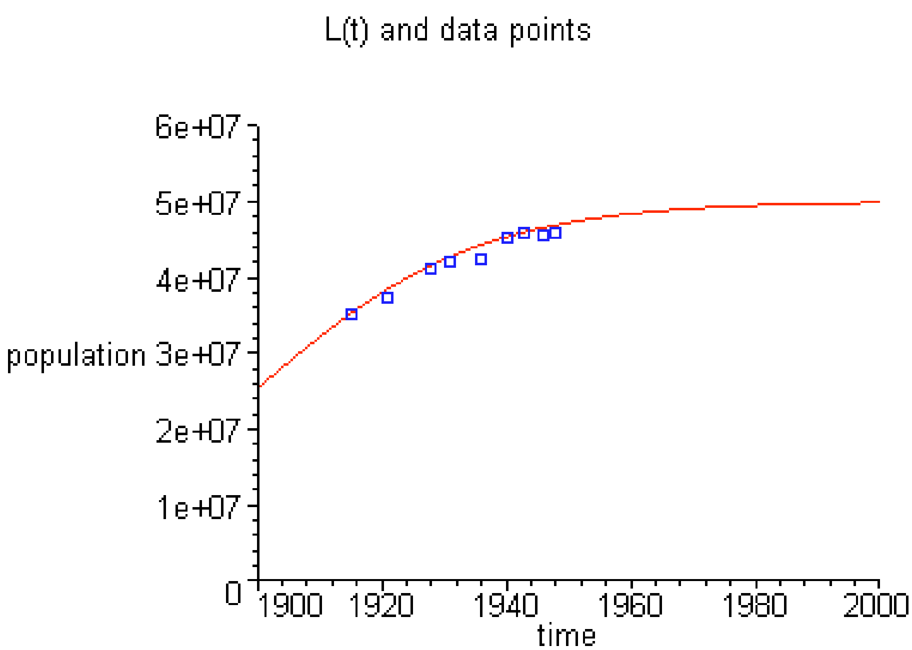

Figure 9. The data points and the model.

The model can now be used to predict the population of Italy in 1960, 1970, and 1980, which we can compare with population data published in the *World Almanac*:

```
> L(1960),L(1970),L(1980);
```

.4837745298 10^8, .4906108639 10^8, .4945989787 10^8

```
> p[9],p[11],p[13];
```

50763000, 53670000, 57040000

Thus, this first model is not growing at a fast enough rate to reflect later population growth. However, the new data from 1970, 1980, and 1990 can be used refine the model. First, the previous 9 data points are graphed with these 3 new points (**Figure 10**):

```
> with(plots):
> newA:=plot([[t[0],p[0]],[t[1],p[1]],[t[2],p[2]],[t[3],p[3]],
[t[4],p[4]], [t[5],p[5]], [t[6],p[6]],[t[7],p[7]],[t[8],p[8]],
[t[9],p[9]],[t[11],p[11]],[t[13],p[13]],[t[14],p[14]]],
x=1900..2000, y=0..60000000, style=point, title=`data points`,
symbol=box, labels=[time, population]):  display(newA);
```

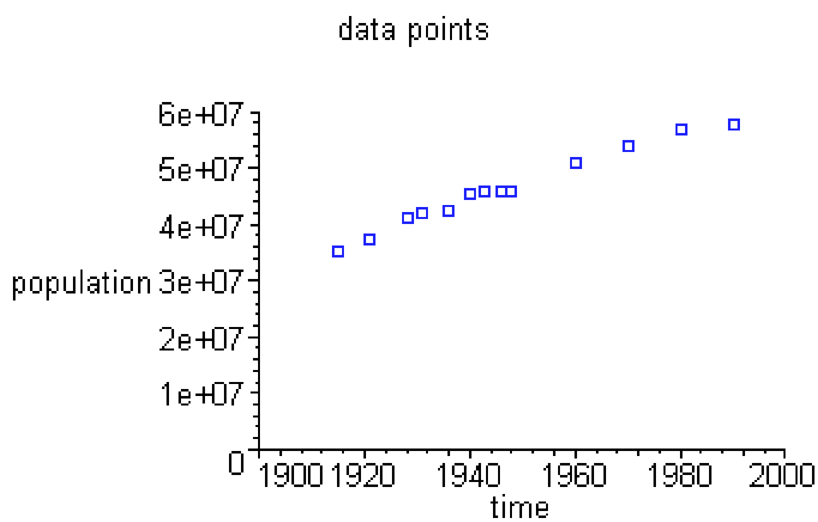

Figure 10. Plot with points for 1970, 1980, and 1990 added.

From this plot, it appears that M should be about 60 million. Estimation of M is very much tied to the history of the country being considered. For Italy, 50 million seemed like a very good estimate for M before 1950, but this changed in the ensuing decades. Using the revised M, the modeling function $L(t)$ becomes:

```
> L:=t-> M/(1+B*exp(-M*k*(t-1915))):M:=60000000:L(t);
```

$$L(t) = \frac{60000000}{1 + Be^{-60000000k(t-1915)}}$$

```
> t[0]:=1915:p[0]:=35240000:solve(p[0]=L(t[0]),B):
> B:=evalf(%);
```

$$B := .7026106697$$

```
> L(t);
```

$$L(t) = \frac{60000000}{1 + .7026106697e^{-60000000k(t-1915)}}$$

Then t_{11} and p_{11} are used to find k. This data point is used since it is the middle point of the three new points being used to update the model.

```
> solve(p[11]=L(t[11]),k);
```

$$.5407883627\ 10^{-9}$$

```
> k:=%:
```

Thus, after simplifying, $L(t)$ looks like:

```
> L(t);
```

$$L(t) = \frac{60000000}{1 + .7026106697 e^{-.03244730176 t + 62.13658287}}$$

The values of $L(t)$ at times t_0 through t_{14} and the actual populations through the year 1990 are:

```
> seq(evalf(L(t[i])), i=0..14);
```

$.3523999999\ 10^8$,$.3801521937\ 10^8$,$.4107307488\ 10^8$,$.4231106310\ 10^8$, $.4426545184\ 10^8$, $.4572499411\ 10^8$,$.4675681357\ 10^8$,$.4773405840\ 10^8$, $.4835519319\ 10^8$, $.5158402926\ 10^8$, $.5269086676\ 10^8$,$.5367000000\ 10^8$, $.5453158102\ 10^8$,$.5528619135\ 10^8$, $.5651651603\ 10^8$

```
> seq(evalf(p[i]), i=0..14);
```

$.35240000\ 10^8$, $.37270493\ 10^8$, $.41168000\ 10^8$, $.42118835\ 10^8$, $.42527561\ 10^8$, $.45330441\ 10^8$, $.45801000\ 10^8$, $.45646000\ 10^8$, $.45706000\ 10^8$, $.50763000\ 10^8$, $.52736000\ 10^8$, $.53670000\ 10^8$, $.55810000\ 10^8$, $.57040000\ 10^8$, $.57657000\ 10^8$

Figure 11 shows how the graph of $L(t)$ fits the data points:

```
> with(plots):
> newBC:=plot(L(t),t=1900..2000,y=0..60000000,
labels=[time,population], color=blue, title=`L(t)`):
display(newA,newBC);
```

Using this new model to predict the population of Italy over the next 50 years gives:

```
> L(2000),L(2010),L(2020),L(2050);
```

$.5744058647\ 10^8$, $.5812764801\ 10^8$, $.5863465534\ 10^8$, $.5947677938\ 10^8$

These model predictions can then be compared to estimates obtained using other projection techniques. For example, the program `IntlPop` uses a version of the cohort method to make population projections (see the Internet sites in Section 5.2 for information about `http://geosim.cs.vt.edu/`, the home page of Project GeoSim at Virginia Tech where this program is available). Using `IntlPop`, the population projections for Italy in the years 2000, 2010, 2020, and 2050 are:

$.58264000\ 10^8$, $.57778000\ 10^8$, $.55498000\ 10^8$, and $.44718000\ 10^8$

There is a great discrepancy between the projections of the logistic model developed above and `IntlPop`. One of the reasons is that the model in the `IntlPop`

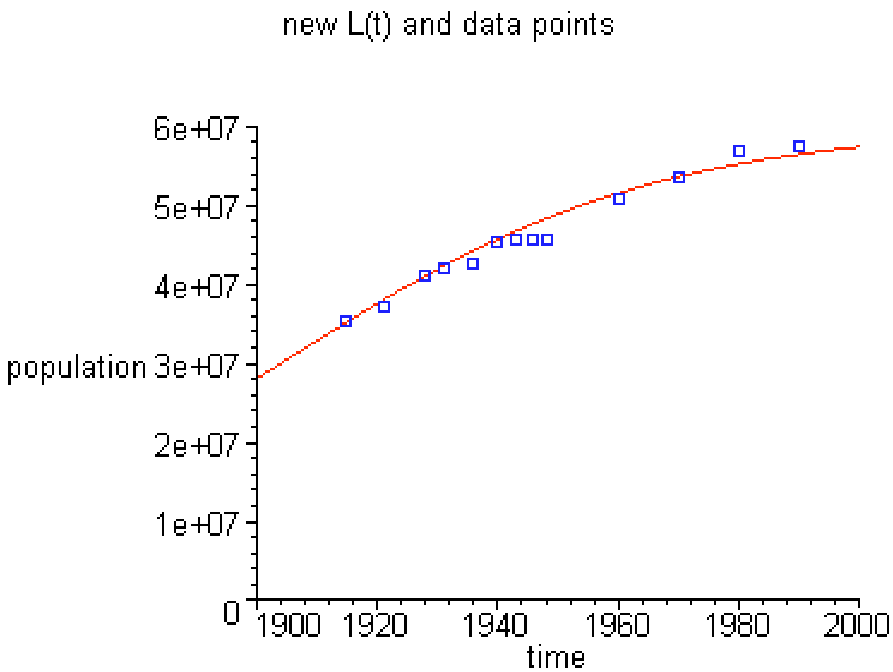

Figure 11. Data points and the improved model.

simulations assumes that the total fertility rate in Italy will remain at the low level of 1.6 (below the replacement rate) over the next half century, contributing to an overall decline in the population. Since the logistic curve is always increasing, it is an inappropriate model for periods of population decline.

This example illustrates some of the difficulties inherent in human population modeling. Exponential and logistic models are very accurate and useful (even for predicting future growth) in a controlled laboratory setting, for getting quite accurate models for backward analysis of human population change, and even for very short-term predictions. However, both models developed in this example failed as long-range predictors of Italy's future population change. The logistic model considers only relations between the overall rate of growth and the carrying capacity of the system; in particular, it could not model the effects of war or the current low fertility rate.

For example, what criteria should be used to evaluate population models? What population characteristics should be incorporated into a model? More-sophisticated models factor in such variables as gender-differentiated birth and death rates, changes due to immigration, and changes within segregated age groups. Is it even possible to develop models that reflect the possibility of war, natural disaster, medical advances, or social change?

To get an idea of some of mathematical methods used in demography (as well as biology and ecology), consult the supplementary resource list in Section 5.3. This includes a listing of some of the pertinent journals, early works in these fields, and reference sources for issues in these areas.

3. Modeling Activities

3.1 Examples with Given Data

The following are "warm-up" exercises in modeling with various functions. The data are provided, as well as questions to guide the analysis.

3.1.1 Exponential Growth

Suppose yeast is cultured at noon on Monday. When the culture is examined at 10:00 A.M. on Tuesday, there are 1.2 thousand organisms present. At 1:45 P.M. on Friday, there are 3.6 thousand organisms. The culture is assumed to grow exponentially.

1. Find a function $P(t)$ for the number of yeast organisms present t hours after noon on Monday (remember to convert hours into decimals).
2. How many organisms were present initially (i.e., at noon on Monday)?
3. What is the growth constant and what does this number mean?
4. How many organisms were there at noon on Wednesday?
5. When will there be 5 thousand organisms present?
6. How fast will the population be growing at that time? How could you estimate this without using the derivative?
7. How long does it take for the population to double in size? How long to quadruple? Is there a pattern here?
8. Graph $P(t)$ and use it to visually check if the answers seem accurate.
9. If another measurement were taken on Saturday, what kind of count would support using an exponential model? What count would suggest using a logistic model?
10. If another measurement were taken on Thursday, what kind of count would support using an exponential model? What count would suggest using a limited growth model?
11. What characteristics of an experiment would contribute to an exponential model being a reasonable choice?

12. What experimental factors might affect how reliable this model is as a predictor of growth after 1:45 P.M. on Friday?

3.1.2 Logistic Growth

Some yeast was grown in a petri dish, and the diameter of one colony was measured, giving the data in **Table 3**. As in the example in Section 2.3, find a logistic curve to model this data, and then answer the following questions:

Table 3.
Yeast data.

date	time	diameter (mm)
9/3	10:00 A.M.	0.75
9/4	9:10 A.M.	1.5
9/5	11:55 A.M.	3
9/7	12:30 P.M.	4
9/9	2:51 P.M.	5.1
9/10	9:30 A.M.	5.25
9/11	10:10 A.M.	5.33
9/12	6:15 A.M.	5.39

1. Estimate the area covered by the colony at 10:25 A.M. on 9/6. How accurate does this estimate seem to be?

2. At what rate is the area increasing at that time? What does this mean about the growth?

3. What is the carrying capacity of the system? What does this represent physically?

4. When does the rate of growth slow down? In other words, when does the population stop growing at an increasing rate and start growing at a decreasing rate? Why does this happen?

5. What is the average area covered by the colony during the course of the experiment? Why might someone be interested in this?

6. Could this model be used to extrapolate to areas after 9/12? For about how long might the predictions be usable?

7. Include a graph of the data points, a graph of the logistic curve, and a graph of the logistic curve with the data points superimposed on it.

3.2 Laboratory Analysis

During the course of this class, data will be collected from several laboratory experiments. For each experiment, a table like the one below can be used to record data. The questions following the table guide the modeling and analysis.

Experiment:

Organism observed:

Measurement method:

Observation #	Date & Time	No. of Hours (in decimals)	Observation

1. Decide what kind of curve (e.g., exponential or logistic) seems to fit data. Justify the choice of the model.

2. Use Maple to get a specific function that models the data. The work should include the following:
   a) a graph of the data points,
   b) work showing how the growth and other constants were determined,
   c) a comparison between the actual data and the values predicted by the modeling function,
   d) a plot of the function, and
   e) a plot of the function with the actual data points plotted on the same graph.

   Note: It may be necessary to try several different values for the constants (e.g. try different data points to solve for the growth constants, adjust your guess for the carrying capacity M, etc.) to get a curve that models the data well.

3. Answer the following questions. These are questions which could not have been answered just from the experimental data but which can be answered using the tools of calculus now that there is a mathematical model for the experiment. Show the calculations used to find the answers to these questions, being sure to include units and comments when appropriate.
   a) What are the constants used, and what do they represent physically?
   b) Estimate the area covered by the colony at 10:00 A.M. on the 5th day of the experiment? How accurate does this estimate seem to be?
   c) At what rate is the growth increasing at that time? What does this mean about the growth?
   d) Was the rate at 10:00 A.M. on the 5th day greater or less than the rate at 10:00 A.M. on the 8th day? What does this mean, and what aspects of the experiment might account for it?

e) Did the rate of growth slow down or change in any way during the course of the experiment? If so, when did the change occur? What might have caused this?

f) What is the average area covered by the colony during the course of the experiment?

g) What would the population be 24 hours after the last observation? What about 48 hours later? For approximately how long might predictions from this model be usable? Why?

h) Comment on your results. How well do they seem to reflect or predict reality?

3.3 Human Populations

After the success in modeling the growth observed in biological experiments, it seems reasonable to use similar modeling techniques to predict human population levels based on current demographic data. This involves the following:

- Creation of human population models for various countries based on recent population levels (to test model accuracy, earlier population data can be used to project figures for years in which census data is available).

- Use of library and Internet resources to gather demographic information (see Sections 5.2 and 5.3 for some useful sources to get started).

- Application of the population simulation program `Intlpop`, which provides such data as recent population levels, life expectancies, fertility rates, infant mortality rates, and net migration data for various countries/regions in the world. Other sites listed in Section 5.2 provide similar population data for various regions in the world and in the United States.

- Discussion of the politics of census figures and population projections, leading to the realization that different sources often provide very different projections (Haub [1987] is helpful).

This project uses the concepts covered to study the original question of human population change.

- Choose a country of interest, and using library or Internet resources, collect as much population data as possible prior to 1950.

- With this data, use Maple to develop a model for the population of that country. Explain the choice of modeling function.

- Use the modeling function to predict the population of that country in 1960, 1970, and 1980.

- Collect data on the actual population of the country in 1960, 1970, and 1980. What might account for any discrepancies? This may require some research into the history of the country.
- Use the data from the 1980s and 1990s to refine the modeling function. If you change the modeling function, explain how and why.
- Compare the actual data with the populations given by the modeling function and seek reasons for any discrepancies. How well does the model estimate the actual populations in 1965 and 1975? Compare function values to actual data if available.
- Use the new modeling function to predict the population of the country for the years 2000, 2010, and 2020.
- Research population literature (including `Intlpop`) to get predictions for the future population of the country. How does the model's predictions compare with those in the literature? What might account for any discrepancies? ("They have a better model" is not an adequate answer—it is important to discuss what factors their model uses that this one does not and to consider differences in the assumptions made.)
- How are human populations different from organisms grown in the lab? How might the modeling functions be modified to give better predictions?
- As an exercise in examining how models can be manipulated to achieve a desired result from political or ideological motives, or even to pander to Hollywood sensationalism, analyze the premise of the movie *Soylent Green*. First, find the population of New York City in 1973 when the movie was made. Then develop a population model supporting the movie's premise that the population of NYC in 2022 will be 40 million. What does this model predict that the population of NYC would have to be today? Find the most recent census data available for the actual population of NYC today and compare it to the value predicted by the model. Comment on the results.

4. Developing Facility with Maple

4.1 Examples and Discussion

The following examples provide practice using Maple and give interpretations of various mathematical concepts in the physical context of modeling population changes. The examples are given in code for Maple V, Release 5. Most of this code should still work with earlier releases, noting that % has replaced " in Release 5. These examples and the following exercises could be modified for use with other computer algebra systems such as Mathematica or Derive, or even with a good graphing calculator.

4.1.1 Graphing

A common use for Maple in this Module is to define a function and plot its graph. Limits for the domain values must be given, but the limits for the range values are optional, as in

```
> f := x-> (x-3)^2 + 4*x - 10;
> plot(f(x),x=-5..5,y=-5..5);
```

To "zoom in" on the graph, simply restrict the x and y ranges, as in

```
> plot(f(x),x=2.3..2.5,y=-0.05..0.05);
```

At this magnification, the curve looks almost linear, which indicates that locally a linear equation (the equation of the tangent line, in fact) gives a good approximation of the function.

Maple can also plot individual data points, as follows:

```
> plot ( [ [1,2], [-3,1], [1.5, 3], [-1,-1], [2,0.5] ], x= -4..3,
y=-2..4, style=point, symbol=diamond);
```

4.1.2 Solving

Maple can be used to solve equations and systems of equations and to find roots of functions. For example, to find the roots of $f(x) = (x-3)^2 + 4x - 10$, use the `solve` command:

```
> solve(f(x)=0,x);
```

This command can be used to determine the time of an event by finding solutions to a given equation. For example, suppose that the concentration of a drug in the blood stream t hours after noon is given by $A(t) = 1.36e^{-0.34t}$. The time at which the concentration of the drug is 0.5 milligrams can be found by:

```
> solve(0.5=1.36*exp(-.34*t));
```

Maple can solve also systems of equations. To illustrate this, suppose that a yeast culture is growing exponentially according to $p(t) = Ce^{kt}$. Assume that 2 days after the experiment begins, there are 140 thousand organisms present; and 3.5 days after, there are 300 thousand present. To find an exponential function $p(t)$ modeling the yeast growth, use

```
> fsolve(140=C*exp(k*2), 300=C*exp(k*3.5), C,k);
```

The difference between the commands `solve` and `fsolve` is that `solve` will return exact solutions, such as $\sqrt{2}$, if possible, while `fsolve` returns floating-point decimals.

4.1.3 Differentiating

Maple differentiates functions using the diff command. The first argument is the function and the second is the independent variable. For example, differentiate $V(r) = \pi r^2 h$ with respect to r:

```
> diff(Pi*r^2*h, r);
```

The diff command can also be used to show that a function is a solution to a differential equation. For example, here's how to show that $y(t) = Ce^{kt}$ is a solution to $y' = ky$:

```
> y:=t->C*exp(k*t):  > simplify (diff(y(t), t)-k*y(t));
```

The result of 0 shows that $y' - ky = 0$ and hence $y' = ky$.

4.1.4 The Shapes of Graphs

The following functions have plots that exhibit the four basic shapes that may occur in a graph.

```
> f1:=x->exp(x):  f2:=x->ln(x):  f3:=x->-ln(x):  f4:=x->20-exp(x):
```

- `> plot(f1(x), x=-3..3);`

  This function is increasing at an increasing rate (increasing and concave up). It might be the shape of a function modeling population growing at an increasing rate, for example bacteria growing with unlimited nutrients.

- `> plot(f2(x), x=-3..3);`

  This function is increasing at a decreasing rate (increasing and concave down). It might represent a population with some limiting factor.

- `> plot(f3(x), x=-1..3);`

  This function is decreasing at an increasing rate (decreasing and concave up), representing a gradually declining population, perhaps one where the birth rate is slightly less than the mortality rate.

- `> plot(f4(x), x=-3..3);`

  This function is decreasing at a decreasing rate (decreasing and concave down). It could model a population experiencing a precipitous drop off in numbers, perhaps following disease or disaster.

4.1.5 Exponential and Logarithmic Functions

Exponential and logarithmic functions are essential to the models developed in this module. An exponential function has the form $y = a^k$, for some fixed constant a. Here are some examples of exponential functions:

```
> plot((1/2)^x, 2^x, x=-5..5);
> plot((1/3)^x, 3^x, x=-3..3);
> plot(seq(i^x, i=2..5), x=-2..2);
> plot(2^x, exp(x), 3^x, x=-3..3);
```

Notice that $y = a^k$ and $y = a^{-k}$ are symmetric about the y-axis and that $y = 0$ is a horizontal asymptote for all such functions. Also, the graph of e^x lies between the graphs of 2^x and 3^x, just as e lies between 2 and 3.

Now consider the graphs of some logarithmic functions. The Maple notation for $\log_b a$ is log[b](a). Recall that $\log_b a = c$ means that $b^c = a$. Also, the natural logarithm is $\ln x = \log_e x$.

```
> plot(log[1/2](x), log[2](x), x=-1..100);
> plot(log[2](x), ln(x), log[3](x), log[4](x), log[10](x), 1,
x=0..10);
```

Notice that $y = \log_a x$ and $y = \log_{a^{-1}} x$ are symmetric about the x-axis and that $y = \log_a x$ intersects the line $y = 1$ at the point $(a, 1)$.

The following graphs illustrate the inverse relationship between $y = a^x$ and $y = \log_a x$ and show that they are reflections of each other about the line $y = x$.

```
> plot(x, 2^x, log[2](x), x=-5..5, y=-5..5);
> plot(x, exp(x), ln(x), x=-5..5, y=-5..5);
```

Plotting a few exponential functions with their derivatives shows that the derivative of $y = a^x$ (which turns out to be $y = a^x \ln a$) is very similar to the exponential function itself, and when $a = e$ they are identical.

```
> f:=(a,x)->a^x:
> g:=(a,x)->diff(f(a,x),x):
> plot(f(2,x),g(2,x), x=-3..3,y=0..8,
title=`2^x and its derivative`);
> plot(f(3,x),g(3,x), x=-3..3, y=0..8,
title=`3^x and its derivative`);
> plot(f(exp(1),x),g(exp(1),x), x=-3..3, y=0..8,
title=`exp(x) and its derivative` );
```

4.2 Maple Exercises

The following problems provide practice using Maple to perform the types of calculations that are likely to arise in this Module.

1. Graphing and zooming

a) Graph $f(x) = e^x$.

b) Graph $g(x) = Ce^{kx}$, where $C = 0.0386$ and $k = 2.18$.

c) Graph $h(t) = \dfrac{M}{1 + Be^{-kMt}}$, where $M = 32$, $B = 21.5$, and $k = 0.18$.

d) Graph $j(x) = 2x^2 - 15x - 10$ from $x = -5$ to 10 and estimate the zeros of $j(x)$ by zooming in.

e) Plot the time and number of people in the computer lab for four different times. Measure times starting with the first observation and converting them from hours/minutes to decimals. Also, use reasonable x and y ranges to display the data. For example, suppose observations at 12:30, 1:00, 1:15, and 1:50 find that there are 10, 8, 3, and 18 people in the lab. The data points would then be [0, 10], [0.5, 8], [0.75, 3], and [1.33,18]. Reasonable x and y ranges for this example would be $x = 0 \ldots 1.5$, $y = 0 \ldots 20$.

2. Solving

a) Solve for x: $x^2 - 4x - 6 = 0$.

b) Solve for x: $ax^2 + bx + c = 0$.

c) Find the roots of $h(x) = x^3 - 3x$ by setting it equal to zero and then solving.

d) If $f(t) = 2.30e^{-0.2t}$ is the concentration of a drug in the blood at time t, find when the concentration will be 2.9 milligrams by finding the roots of $f(t) - 2.9$. Then determine when the concentration of the drug will be 1.3 milligrams, and give possible interpretations of the solution.

e) Solve the system of equations: $y = 3x - 5$, $3y - 2x = 12$.

f) Suppose a bacterial culture is growing exponentially, 200 thousand bacteria are present 1 hour after the experiment begins, and 265 thousand are present after 2.5 hours. Find the constants C and k by solving $p(1) = 200$ and $p(2.5) = 265$. for C and k, where $p(t) = Ce^{kt}$.

g) Unfortunately, when very large or small numbers are involved, Maple (or any other CAS or a graphing calculator) may give an error message. For example, if solving $p(1) = 200$ and $p(26) = 950$ for C and k, where $p(t) = Ce^{kt}$, yields an error message using the technique above, it is still possible to solve for k by using the fact that for exponential functions the growth constant is equal to

$$\frac{\ln A_1 - \ln A_2}{t_1 - t_2},$$

where A_1 is the amount at time t_1 and A_2 is the amount at time t_2. Thus, it is possible to find k first and then solve for C. Carry out this computation.

3. Differentiation

a) Differentiate $f(x) = x^3 - x^2$.
b) Differentiate $g(t) = CXe^{kt}$ with respect to t.

4. Differential Equations

a) Show that $y = CXe^{kt}$ satisfies the differential equation $y' = ky$.

b) Show that $v(t) = M\left(1 - e^{-kt}\right)$ is a solution to the differential equation $v' = k(M - v)$.

c) Show that $y(t) = M/\left(1 + Be^{-Mkt}\right)$ satisfies the differential equation $y' = ky(M - y)$.

5. Shapes of Graphs
Plot the graphs of the functions below, adjusting the y-range as needed to get a good picture. Although these graphs are unlikely to represent models of population growth, consider what their shapes would indicate about the way the population would be changing. What might the concavity, asymptotes, inflection points, and relative extrema indicate about the behavior of a population?

a) $f(x) = 10x(x-0.5)(x+1.5)(x-1)(x+0.5)(x+0.75)$, for $-1.75 \le x \le 1.5$.

b) $f(x) = \dfrac{x^2 - 4}{(x+1)(x-3)}$, for $-6 \le x \le 6$.

c) $f(x) = x^3 - x^2$, for $-1 \le x \le 1$.

5. Resources

5.1 Laboratory Notes

Below is the basic information for each lab that can be included in this Module. For each experiment, the necessary equipment is listed and the procedure is specified. If there is not time or facilities to do all of the experiments, it suffices to choose one example of exponential growth and one of logistic growth.

The populations studied include slime molds (to model qualitatively exponential growth), flour beetles (to model exponential growth), yeast cultures (to model both exponential and logistic growth), and ciliated protistans (to model exponential and logistic growth).

Some materials can be obtained at a grocery store. Other items can be ordered from biological supply companies, such as the following:

Carolina Biological Supply Company
2700 York Road
Burlington, NC 27215
(800) 334–5551

Connecticut Valley Biological
82 Valley Road
P.O. Box 326
Southampton, MA 01073
(413) 527–4030

5.1.1 Exponential Growth with Slime Molds and Flour Beetles

Equipment needed (for slime mold experiment)
Slime molds (e.g., *Physarum polycephalum* or *Dictyostelium discoideum*)
Non-nutrient agar (2% agar)
Oatmeal
Petri dishes

Slime Mold Procedure:
The slime molds are grown in a petri dish on sterile agar with oatmeal sprinkled on top. When the slime molds run out of oatmeal, they will move out in search of food. The actual timing of this "migration" will depend on the initial amount of food (this could even become the basis of an experiment, if so desired). It will most likely take at least 2 weeks for the slime molds to begin migrating.

Flour Beetle Information:
The experiment with flour beetles is outlined in Part B of Glase and Zimmerman [1993]. Further descriptions of this experiment and the population changes being modeled can be found in Glase and Zimmerman [1992]. The beetles provide a good example of exponential growth, but it takes upwards of 7 weeks to collect sufficient data.

5.1.2 Exponential/Logistic Growth with a Yeast Culture

A yeast culture exhibits exponential growth during the first 10 to 36 hours of the experiment and logistic growth if the experiment continues for 48 hours.

Equipment needed
Culture tubes containing 15 ml of YPD broth (consisting of 1% yeast extract, 2% peptone, and 2% dextrose)
Counting chamber (a spectrophotometer can also be used)
Microscope (400X)
Yeast culture (fry yeast available in grocery stores can be used)
Orbital shaker

Yeast Procedure:
Add one grain of dried yeast to 15 ml of sterile YPD in a culture tube. Gently dissolve the yeast grain. Incubate the tube at room temperature, or for faster growth at 30°C. Periodically (and at least at the beginning of the experiment), remove 10 μl of broth and place on the counting chamber. At room temperature,

observations should be made every 4 hours or so. Count the number of cells to determine the concentration of yeast cells. The small squares of the counting chamber have an area of 1/400 mm^2. The depth of the liquid is 1/50 mm. Cell concentration is reported as cells/ml. If a spectrophotometer is used, then the absorbance of the culture can be measured at 600 nm.

5.1.3 Logistic Growth with a Bacteria Culture

This is perhaps the least complicated of all the experiments. In it, bacterial cultures get restricted nutrients and exhibit logistic growth. The amount of time needed to see the logistic growth will depend upon the incubation temperature of the agar plates. At 35°C, it could take 24 to 36 hours, whereas at room temperature it may require several days.

Equipment needed
Bacteria culture (*Bacillus subtilis*)
Tryptic soy agar plate (DIFCO)
Ruler
Magnifying glass

Bacteria Procedure:
Streak out *Bacillus subtilis* on a tryptic soy agar plate (DIFCO) to obtain well-separated colonies. To measure colony growth accurately, it may be most effective to grow a single culture on a single plate. The plates can be incubated at either room temperature or 35°C, depending on the desired rate of growth. At room temperature, observations should be made about once a day. The diameters of the resulting bacterial colonies can be measured with a ruler and magnifying glass.

5.1.4 Population Crash with Ciliated Protistans

This experiment demonstrates exponential and logistic growth followed by a precipitous decline or "crash" in population using a few different species of ciliated protistans.

Equipment needed
Handout entitled "Population Ecology: Experiments with Protistans"
Protistan cultures
Stereoscopic binocular microscope
Pasteur pipettes and bulbs
Volumetric pipettes and dispensers
Depression slides and counting plates or hemocytometer
Sterile spring water
Culture vials and plugs
Concentrated liquid food

Protistan Procedure:
The experimental procedure is outlined in detail in Part A of "Population Ecology" (see Section 5.3). Four to five different protistan species are available for selection. One type containing photosynthetic green algae (*Paramecium bursarie*) is useful in experiments of light vs. dark growth. For the purposes of this Module, it may be most worthwhile to simplify the parameters of the experiment as much as possible and to set up only single species cultures. The "Population Ecology" handout discusses more complicated variations of a single species study, such as predator/prey and competition experiments, which may be worthwhile if time permits. Protistan cultures are grown in culture vials, and pp. 42–43 of "Population Ecology" gives precise and detailed techniques for counting individual organisms. For example, after thoroughly mixing the vials, Pasteur pipettes are used to place sample of the cultures onto counting plates for microscopic inspection. Using these methods, protistan growth can be closely monitored and then modeled.

5.2 Population Internet Sites

`http://www.census.gov/`
Home page of the U.S. Census Bureau
This site offers a rich collection of social, demographic, and economic information. For example, one can search for population information indexed by word or by place, or on a clickable map. Using a cohort-component model, the Census Bureau offers population projections from 1996 to 2050 for U.S. resident population based on age, race, sex, and Hispanic origin. This site also has population clocks for the U.S. and world as well as information about Census 2000 in the U.S.

`http://geosim.cs.vt.edu/`
Home page of Project GeoSim at Virginia Tech
Project GeoSim offers many educational modules for introductory geography courses, including `HumPop` and `IntlPop`. `HumPop` is a multimedia tutorial program introducing and illustrating population concepts and issues, while `IntlPop` is a useful population simulation program for various countries and regions around the world.

`http://www.nidi.nl/links/nidi6000.html`
Home page of NiDi
Maintained by the Netherlands Interdisciplinary Demographic Institute, this site offers a comprehensive overview of demographic resources on the Internet. It contains about 400 external links to various useful sites about software and demographic models, census/survey and data facilities, research institutes and organizations, literature, conferences, and other information resources.

`http://www.popnet.org/` Home page of Popnet
Created by the Population Reference Bureau, Popnet offers links to many sites

containing global population information. For example, it contains links to organizational sources in many countries throughout the world, offers various demographic statistics, and includes a "demographic news update." It also contains a clickable world map through which the user can obtain a directory of Websites that provide region specific information.

`http://www.prb.org/`
Home page of the Population Reference Bureau
This site contains useful information on U.S. and international population trends. For example, it offers the World Population Data Sheet, an annual publication containing recent population estimates, projections, and key indicators for areas with populations of 150,000 or more as well as all members of the United Nations. This site also links to reports on a variety of population issues.

`http://www.iisd.ca/linkages/`
Home page of Linkages
Provided by the International Institute for Sustainable Development, Linkages is a clearing-house for information on previous and upcoming international meetings related to development and environment. For example, it contains a link (`http://www.iisd.ca/linkages/cairo.html`) to the home page of the 1994 Cairo International Conference on Population and Development as well as links to current conferences on population.

`http://popindex.princeton.edu/`
Home page of the Population Index
This site contains an online version of the Population Index, a primary reference tool offering an annotated bibliography of recently published journal articles, books, working papers, and other materials on population topics. The site has an on-line database (for 1986–1999) which can be searched by author, geographical region, subject matter, and year of publication.

`http://coombs.anu.edu.au/ResFacilities/DemographyPage.html`
Home page of Demography and Population Studies at ANU
Provided by Australian National University, this regularly updated site has an assortment of links to demographic information centers and facilities worldwide. These include links to various demography and population studies WWW servers, databases of interest to demographers, and demography and population studies gopher servers.

`http://www.popassoc.org/`
Home page of the Population Association of America
This site offers information about the Population Association of America and its quarterly journal Demography. Articles from this journal can be accessed using the full text archive JSTOR (which is linked to this site at `http://www.`

`jstor.org/jstor/`), through which articles from other pertinent journals also can be obtained.

5.3 Additional Resources

5.3.1 Early Works

Gause, G.F. 1934. *The Struggle for Existence*. Baltimore, MD: Williams and Wilkins.

Lotka, A.J. 1925. *Elements of Physical Biology*. Baltimore, MD: Williams and Wilkins. 1956. Reprinted as *Elements of Mathematical Biology*. New York: Dover.

Malthus, T.R. 1798. *An Essay on the Principle of Population As It Affects the Future Improvement of Society*. London, England: J. Johnson.

Pearl, Raymond, and L.J. Reed. 1920. On the rate of growth of the population of the United States since 1790 and its mathematical representation. *Proceedings of the National Academy of Science* 6: 275–288.

Verhulst, P.F. 1938. Notice sur la loi que la population suit dans son accroissement [Note on the law followed by a growing population]. *Correspondance Mathématique et Physique* (Brussels) 10: 113–121.

Volterra, Vito. 1926. Variations and fluctuations in the number of individuals of animal species living together. Appendix in *Animal Ecology*, edited by R.N. Chapman, 409–448. New York: McGraw-Hill.

5.3.2 Biology

Krebs, C.J. 1972. *Ecology: The Experimental Analysis of Distribution and Abundance*. New York: Harper & Row. A good review of classical population experiments of the first half of the twentieth century on yeast, bacteria, *Drosophila*, and flour beetles is provided on pp. 190–200.

May, R.M., ed. 1976. *Theoretical Ecology Principles and Applications*. Philadelphia, PA: W.B. Saunders.

Seeley, H.W., and P.J. VanDemark. 1981. *Microbes in Action*, 82–83. San Francisco, CA: W.H Freeman.

5.3.3 Demography

Keyfitz, Nathan. 1968. *Introduction to the Mathematics of Population*. Reading, MA: Addison-Wesley. 1977. 2nd ed.

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Shryock, H.S., J.S. Siegel, and associates. 1973. *The Methods and Materials of Demography*, vol. 2. Washington, DC: U.S. Bureau of the Census.

Wattenberg, Ben. 1991. *The First Universal Nation*. New York: Free Press.

The World Almanac and Book of Facts. 1916 and 1924 eds. New York: New York World. 1931, 1933, 1938, 1942, 1946, 1948, 1950 eds. New York: New York World-Telegram 1961 ed. New York: New York World-Telegram & The Sun. 1967, 1972, 1977, 1982 eds. New York: Newspaper Enterprise Association. 1991 ed. New York: Pharos Books.

Curve Fitting and the Logistic Model

Cavallini, Fabio. 1993. Fitting a logistic curve to data. *College Mathematics Journal* 24 (3) (May 1993): 247–253.

Macchetti, C., P. Meyer, P., and J. Ausubel. 1996. Human population dynamics revisited with the logistic model: How much can be modeled and predicted? *Technological Forecasting and Social Change* 52 (1) (1 May 1996): 1-30.

Maruszewski, R. F. 1994. Approximating the parameters for the logistic model. *Mathematics and Computer Education* 28 (1) (Winter 1994): 16–19.

Matthews, J.H. 1992. Bounded population growth: A curve fitting lesson. *Mathematics and Computer Education* 26 (2) (Spring 1992): 169–176.

A Few Pertinent Journals

Demography

Ecology

Journal of the American Statistical Association

Journal of Ecology

Population and Development Review

Population Index

Population Studies

6. Teaching Notes

Ideally, this Module would run throughout a first-semester calculus course. Some lab experiments take several weeks to develop growth patterns, and these should be started at the beginning of the course. Others take as little as a few days to complete. Meaningful discussion of the data from these labs can begin as soon as students can graph data points and functions. More in-depth analysis can continue after the definition of the derivative has been developed and various properties and examples of derivatives have been considered. The Module is designed to give a context for and hands-on experience with central concepts such as derivatives, exponential and logarithmic functions, properties of graphs, and integration in roughly the order that they are introduced in a standard elementary calculus course.

The following activities should be done early in the course, both to introduce the module and to leave enough time to collect data:

- Distribution of scenes from the science fiction movie *Soylent Green* to begin discussion of motivating sociological and political issues surrounding population changes.

- Assignment of background reading, perhaps containing dramatically different population projections, to illustrate some of the difficulties in forming accurate demographic predictions (see Alonso and Starr [1987], for example).

- Initialization of the experiments. Some of the biology experiments can take up to several weeks to run. It is important to set them up at the beginning so that students can collect data for their final projects throughout the course.

- Introduction to the Maple CAS. To do the exercises associated with these projects, students need to be able to use Maple to perform basic equation and function manipulations. In particular, they must be able to solve equations and systems of equations and to find the roots of functions. They must be able to differentiate and to verify solutions of differential equations. They must also be able to plot points and graph functions. Section 4, **Developing Facility with Maple**, is designed to help students to develop these skills.

- Practice modeling using the examples with given data in Section 3.1.

A version of the projects in this Module that is not Maple-specific and includes classroom demonstrations and student handouts is available [Ashline and Ellis-Monaghan 19???].. There is also an expository paper describing the *need date* authors' experiences with these projects [Ashline and Ellis-Monaghan 1999].

References

Alonso, William, and Paul Starr, eds. 1987. *The Politics of Numbers*. New York: Russell Sage Foundation.

need date Ashline, George, and Joanna Ellis-Monaghan. 19???. How many people are in your future? Elementary models of population growth. In *Making Meaning: Integrating Science Through the Case Study Approach to Teaching and Learning*, ???–???. . New York: McGraw-Hill Primis. *need pp*

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Cohen, Joel. 1995. *How Many People Can the Earth Support?* New York: W.W. Norton.

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Fleisher, Richard, director. 1973. *Soylent Green*. With Charleton Heston, Leigh Taylor-Young, Chuck Connors, Joseph Cotton, Edward G. Robinson. MGM. Available on videocassette.

Glase, J.C., and Melvin Zimmerman. 1992. Studies in protozoan population ecology. In *Tested Studies for Laboratory Teaching, Proceedings of the Thirteenth Workshop/Conference of the Association for Biology Laboratory Education (ABLE)*, 19–66.

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Harrison, Harry. 1966. *Make Room! Make Room!* Boston, MA: Gregg Press.

Haub, Carl. 1987. Understanding population projections. *Population Bulletin* 42 (4) (December 1987). *need pp*

Population Reference Bureau. 1999. *1999 World Population Data Sheet*.

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Tapinos, Georges, and Phyllis T. Piotrow. 1980. *Six Billion People.* New York: McGraw-Hill.

Wattenberg, Ben. 1987. *The Birth Dearth*. New York: Ballantine Books.

Acknowledgments

We would like to thank the following St. Michael's College professors for so enthusiastically sharing the resources of their respective fields with us: Glenn Bauer and Peter Hope (Biology), Vincent Bolduc (Sociology), and Richard Kujawa (Geography).

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