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The dissipative Budden problem: Effect of converted-wave damping on primary-wave reflection

A.N. Kaufman [a], E.R. Tracy [b], J.J. Morehead [a], A.J. Brizard [a]

[a] *Lawrence Berkeley National Laboratory and Physics Department, University of California, Berkeley, CA 94720, USA*

[b] *Physics Department, College of Wm. & Mary, Williamsburg, VA 23187-8795, USA*

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Abstract

The Budden equation represents a double conversion, whereby a primary wave first converts to a localized secondary wave, which then converts to a reflected primary wave. We analyze this process in phase space, to include secondary-wave damping between the two conversions, which reduces the reflection coefficient. The results are then applied to gyroresonant damping of the (secondary) ion-hybrid wave by fusion alpha-particles, suggesting a diagnostic test. © 1999 Elsevier Science B.V.

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The *Budden equation* [1] describes the one-dimensional propagation of a primary wave a through a nonuniform medium which also supports a spatially localized secondary wave b at position $x = x_b$ (say). The *resonant coupling* between the waves causes a fraction C of the energy flux of the primary wave to be *converted* to that of the secondary wave. In addition, a fraction R of the primary wave's energy flux is *reflected*, and the remainder $T = 1 - R - C$ is *transmitted* through the resonance layer. In this paper, we study the effect on the primary wave of the secondary wave damping. By using a ray phase-space analysis [2], we show that the conversion is a *two-step* process, with transmission occurring at the first step, reflection at the second step, and secondary-wave damping being effective between the two steps (Fig. 1).

First, we derive the coupled equations for the two wave fields, $a(x)$ and $b(x)$, first in the absence of dissipation and then in its presence. We present a geometric interpretation of the conversion process, yielding an explicit formula for the effect of damping on reflection. We then solve the equations analytically to verify the result. Next, we apply these results to the conversion of a magnetosonic wave a to an ion-hybrid wave b in a slab model of tokamak geometry [3]. In a fusion plasma, the neonatal alphas interact gyroresonantly with the ion-hybrid wave b, but not with the right-circularly-polarized magnetosonic wave a. We evaluate the damping coefficient and the resulting modification of reflection, and suggest a diagnostic test. Finally, we summarize our results, and discuss generalizations.

For linear wave propagation in a medium with spatial variation in only one coordinate (x), we express the N-component wave field as

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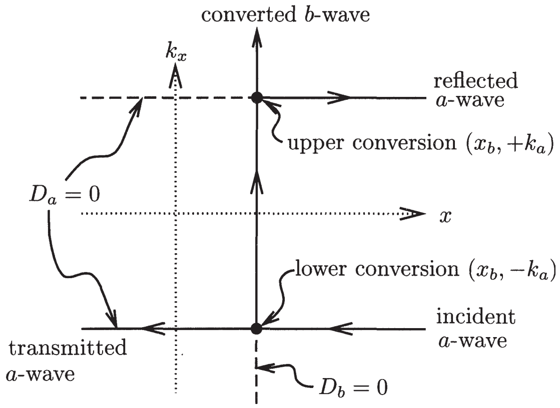

Fig. 1. In 2-dimensional phase space (x, k_x), the dispersion surfaces, $D_a = 0$ and $D_b = 0$, are respectively the horizontal lines $k_x = \pm k_x$ and the vertical line $x = x_b$. Their intersections are the lower and upper conversion points. The a-rays and b-rays are coincident with the dispersion surfaces in 2D phase space; the direction of energy flux, or group velocity, is indicated by arrows. The lower conversion is from incident a-wave to intermediate b-wave; the upper is from the latter to the reflected a-wave.

$$\boldsymbol{\psi}(x) \exp[\mathrm{i}(k_y y + k_z z - \omega t)] , \tag{1}$$

with k_y, k_z, ω as constants. The underlying physics leads to a linear ordinary differential equation for $\boldsymbol{\psi}(x)$,

$$\mathsf{D}(\hat{k}_x, x; k_y, k_z, \omega) \cdot \boldsymbol{\psi}(x) = 0 , \tag{2}$$

where D is an $N \times N$ matrix, whose elements are given functions of its arguments, and $\hat{k}_x \equiv -\mathrm{i}\partial_x$ is an operator on $\boldsymbol{\psi}(x)$. In our plasma application, $\boldsymbol{\psi}$ is the 3-component electric field, and the dispersion matrix D, obtained from the linearized Vlasov or fluid equations, plus the Maxwell equations, can be obtained explicitly.

In the *absence* of dissipation, we can replace (2) by a variational principle; the stationarity of the functional

$$\mathcal{A}(\boldsymbol{\psi}) \equiv \int \mathrm{d}x\, \boldsymbol{\psi}^*(x) \cdot \mathsf{D} \cdot \boldsymbol{\psi}(x) , \tag{3}$$

with respect to infinitesimal variations of $\boldsymbol{\psi}(x)$ yields (2), if the operator D is Hermitian. For a *mode-coupling* problem, we constrain the wave field $\boldsymbol{\psi}(x)$ to represent the superposition of the two modes a and b:

$$\boldsymbol{\psi}(x) = \hat{\boldsymbol{e}}_a a(x) + \hat{\boldsymbol{e}}_b b(x) . \tag{4}$$

The two polarizations $(\hat{\boldsymbol{e}}_a, \hat{\boldsymbol{e}}_b)$ are obtained from (2) in regions of phase space where the coupling is negligible. On substituting (4) into (3), the variational principle reduces to

$$\mathcal{A}(a, b) = \int \mathrm{d}x\, [a^* D_a a + b^* D_b b + a^* \eta b + b^* \eta^\dagger a] , \tag{5}$$

where the dispersion operators D_j $(j = a, b)$ are projections of D onto the polarizations,

$$D_j \equiv \hat{\boldsymbol{e}}_j^* \cdot \mathsf{D} \cdot \hat{\boldsymbol{e}}_j , \tag{6}$$

and the coupling operator η is its cross-projection:

$$\eta \equiv \hat{\boldsymbol{e}}_a^* \cdot \mathsf{D} \cdot \hat{\boldsymbol{e}}_b , \quad \eta^\dagger \equiv \hat{\boldsymbol{e}}_b^* \cdot \mathsf{D} \cdot \hat{\boldsymbol{e}}_a . \tag{7}$$

On varying $\mathcal{A}$ with respect to $a(x)$ and $b(x)$, we obtain the coupled equations for the two wave fields,

$$D_a a + \eta b = 0 , \quad \eta^\dagger a + D_b b = 0 . \tag{8}$$

In the Budden problem, the primary wave a has a dispersion function of the form

$$D_a(k_x) = \kappa_a [k_a^2(k_y, k_z, \omega) - k_x^2] , \tag{9}$$

quadratic in k_x and *independent* of x. The secondary wave b, on the other hand, has a dispersion function *independent* of k_x and *linear* in x,

$$D_b(x) = \kappa_b [x - x_b(\omega)] . \tag{10}$$

(Below we derive these forms, and determine the constants κ_a, κ_b and the functions k_a, x_b.)

In the *absence* of coupling $(\eta \to 0)$, the uncoupled Eqs. (8) become

$$D_a a = 0 , \tag{11}$$

$$D_b b = 0 . \tag{12}$$

The primary wave field $a(x)$ is, from (11) and (9), a superposition,

$$a(x) = a_- \mathrm{e}^{-\mathrm{i}k_a x} + a_+ \mathrm{e}^{+\mathrm{i}k_a x} , \tag{13}$$

of *left-going* and *right-going* waves, with phase-velocities $\mp(\omega/k_a)$. (If mode a is non-dispersive, i.e., $\mathrm{d}k_a/\mathrm{d}\omega = k_a/\omega$, these are also the group velocities, $\dot{x}$.) For mode b, we Fourier-transform to k_x-space:

$$b(k_x) \equiv \int \mathrm{d}x\, b(x)\, \mathrm{e}^{-\mathrm{i}k_x x} , \tag{14}$$

so that, in (10), x becomes the operator $\hat{x} \equiv +\mathrm{i}\partial/\partial k_x$. Then (12) yields

$$b(k_x) = b_0 \mathrm{e}^{-\mathrm{i}x_b k_x}\,, \tag{15}$$

a wave propagating in k_x-space. In x-space, (15) is

$$b(x) = 2\pi b_0 \delta(x - x_b)\,, \tag{16}$$

showing that the secondary wave is localized on the ray $x = x_b$. (The group velocity in k_x-space is $\dot{k}_x = -\partial\omega(x)/\partial x = -[\mathrm{d}x_b/\mathrm{d}\omega]^{-1}$; it is positive in our application.)

As is evident from Fig. 1, the two modes are *resonant* at the *lower* ($k_x = -k_a$, $x = x_b$) and *upper* ($k_x = +k_a$, $x = x_b$) crossings of their rays. In the Budden problem, the coupling η is treated as a constant (determined locally from (7)). If wave energy is *incident* only on the a_- branch, we see that there are three possible ray paths: (1) *transmission* through the lower crossing ℓ; (2) first conversion from a_- to b at ℓ, followed by transmission at u, leading to net *conversion*; (3) first conversion from a_- to b at ℓ, followed by a second conversion to a_+ at u, leading to *reflection*. Let T_ℓ be the transmission coefficient at the lower crossing; i.e., the ratio of energy flux transmitted to energy flux incident. Since energy conservation can be formulated in phase space [4], we can define the lower conversion coefficient C_ℓ, such that $C_\ell + T_\ell = 1$. At the upper crossing, the secondary wave b is *incident* with relative energy flux C_ℓ; thus the net conversion is $C = T_u C_\ell$, and the reflection is $R = C_u C_\ell$. Because both crossings in our problem have the same parameters, we have $T_u = T_\ell$ and $C_u = C_\ell$. Further, $T = T_\ell$, and so $R = (1-T)^2$ and $C = T(1-T)$. Checking, we note that $R + C = 1 - T$, as required.

To evaluate T, we use the standard formula for linear conversion [5–7]

$$T = \exp[-2\pi|\eta|^2/\mathcal{B}]\,, \tag{17}$$

where the Poisson bracket $\mathcal{B}$ is

$$\begin{aligned}\mathcal{B} &\equiv |\{D_a, D_b\}| = |(\partial D_a/\partial k_x)(\partial D_b/\partial x)| \\ &= 2k_a\kappa_a\kappa_b\,.\end{aligned} \tag{18}$$

Thus

$$T = \exp[-\pi k_a L_0]\,, \tag{19}$$

where

$$L_0 \equiv |\eta|^2/k_a^2\kappa_a\kappa_b\,. \tag{20}$$

Now we wish to relate our approach to the conventional Budden equation. In (8), we eliminate b, and use (9) for D_a and (10) for D_b. Simple algebra yields the Budden equation for $a(x)$,

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \left(1 - \frac{L_0}{x - x_b}\right)k_a^2\right]a(x) = 0\,. \tag{21}$$

In terms of a WKB representation ($\mathrm{d}/\mathrm{d}x \to \mathrm{i}k_x$), this equation states that

$$k_x^2(x) = k_a^2\left(1 - \frac{L_0}{x - x_b}\right)\,. \tag{22}$$

It thus exhibits a "resonance" $k_x^2 \to \infty$ at x_b, and a "cutoff" $k_x^2 = 0$ at $x_b + L_0$. Eq. (19) for T, and the relation $R = (1-T)^2$ are in agreement with the traditional [1] analytic solution of (21).

We now include the effects of dissipation, by allowing D in (2) to have an additional anti-Hermitian part $\mathrm{i}\mathsf{D}''$; i.e., $\mathsf{D} \to \mathsf{D}' + \mathrm{i}\mathsf{D}''$. In place of a variational principle, we insert (4) directly into (2), as a Galerkin approximation, and then multiply by $\hat{\boldsymbol{e}}_a^*$ and by $\hat{\boldsymbol{e}}_b^*$. We again obtain (8), but now D_a and D_b may be complex-valued functions of their arguments, while $\eta^\dagger = \hat{\boldsymbol{e}}_b^* \cdot \mathsf{D} \cdot \hat{\boldsymbol{e}}_a$. (In the Budden problem, we have $\eta^\dagger = \eta^*$, but this need not be true in general.)

In our plasma application, dissipation D'' affects only mode b directly; i.e., $\hat{\boldsymbol{e}}_a^* \cdot \mathsf{D}'' = 0 = \mathsf{D}'' \cdot \hat{\boldsymbol{e}}_a$. Thus D_a and η are unchanged, while the complex D_b can be expressed as

$$D_b(x, k_x) = \kappa_b[x - x_b + \mathrm{i}\nu]\,. \tag{23}$$

Generically, ν could depend on both x and k_x. To proceed analytically, we neglect its x-dependence by evaluating it at the hybrid-wave's location x_b; i.e., $\nu(x, k_x) \to \nu(x_b, k_x) \to \nu(k_x = -\mathrm{i}\mathrm{d}/\mathrm{d}x)$. In the Budden equation (21), the term $L_0/(x - x_b)$ is now replaced by $L_0/[x - x_b + \mathrm{i}\nu(-\mathrm{i}\mathrm{d}/\mathrm{d}x)]$. While daunting in the x-representation, this dissipative Budden equation is easy to solve in the k_x-representation.

First let us consider the effect of $\nu(k_x)$ *heuristically*. The *uncoupled* Eq. (12) now reads, in the k_x-representation,

$$\left[\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}k_x} - x_b + \mathrm{i}\nu(k_x)\right] b(k_x) = 0\,, \tag{24}$$

with the solution (normalized by $b(k_x = 0) = 1$)

$$b(k_x; \eta = 0) = \mathrm{e}^{-\mathrm{i}x_b k_x} \exp\left[- \int_0^{k_x} \mathrm{d}k'_x\, \nu(k'_x) \right]. \tag{25}$$

Thus $\nu(k_x)\,\mathrm{d}k_x$ is the exponential damping in the interval $\mathrm{d}k_x$. Hence we expect that, *between* the two conversions (see Fig. 1), the b-wave undergoes the amplitude damping $\mathrm{e}^{-\Gamma}$, with

$$\Gamma \equiv \int_{-k_a}^{k_a} \mathrm{d}k_x\, \nu(k_x)\,. \tag{26}$$

As a result, the energy entering the second conversion is reduced by $\mathrm{e}^{-2\Gamma}$, and accordingly the reflection and net conversion coefficients are reduced by this factor.

With the insight provided by the heuristic argument, we now return to the analytic calculation. We solve (8) for $a(k_x)$ using (9),

$$a(k_x) = -\frac{\eta}{\kappa_a (k_a^2 - k_x^2)}\, b(k_x)\,. \tag{27}$$

Then (8) and (23) yield for $b(k_x)$:

$$\left[\frac{\mathrm{d}}{\mathrm{d}k_x} + \nu(k_x) + \mathrm{i}x_b\right] b(k_x) = \frac{-\mathrm{i}k_a^2 L_0}{k_a^2 - k_x^2}\, b\,, \tag{28}$$

with solution

$$b(k_x) = b(k_x; \eta = 0) \left| \frac{k_x - k_a}{k_x + k_a} \right|^{\mathrm{i}k_a L_0/2} \times \begin{cases} 0, & k_x < -k_a, \\ b_1, & -k_a < k_x < k_a, \\ b_2, & k_x > k_a. \end{cases} \tag{29}$$

The condition of no incoming b-wave implies that $b(k_x) = 0$ for $k_x < -k_a$. In the other two intervals, b_1 and b_2 are the integration constants.

Inserting (29) into (27), and Fourier-transforming, we obtain

$$a(x) = -\frac{\eta}{\kappa_a} \int \frac{\mathrm{d}k_x}{2\pi}\, \mathrm{e}^{\mathrm{i}k_x x}\, \frac{b(k_x)}{k_a^2 - k_x^2}\,. \tag{30}$$

To obtain the transmission and reflection coefficients, we examine the asymptotics of $a(x)$ as $x \to \pm\infty$. Then only the neighborhoods of the branch points at $\pm k_a$ contribute, and we find the form (13), with

$$\left|\frac{a_+}{a_-}\right|(x) = \left| \frac{b_1 J^*(x; -\mu) - b_2 J(x; \mu)}{b_1 J(x; -\mu)} \right| \mathrm{e}^{-\Gamma} \tag{31}$$

and

$$T \equiv \left| \frac{a_-(-\infty)}{a_-(+\infty)} \right|^2 = \left| \frac{J(x \to -\infty; -\mu)}{J(x \to +\infty; -\mu)} \right|^2, \tag{32}$$

where

$$J(x; \mu) \equiv \int_0^\infty \frac{\mathrm{d}k}{k}\, \mathrm{e}^{\mathrm{i}kx} k^{\mathrm{i}\mu} \quad (\mu \equiv \tfrac{1}{2} k_a L_0) \tag{33}$$

and we note that

$$J^*(x; \mu) = J(-x; -\mu)\,. \tag{34}$$

(We do not need to use the fact that (33) can be expressed in terms of a gamma function.) The boundary condition, of no incident a-wave from the left, is $a_+(x \to -\infty) = 0$; so (31) yields

$$b_2 J(x \to -\infty; \mu) = b_1 J^*(x \to -\infty; -\mu) \tag{35}$$

as the relation between the integration constants. Then the reflection is

$$R \equiv \left| \frac{a_+}{a_-} \right|^2 (x \to +\infty) = \mathrm{e}^{-2\Gamma} \left| 1 - \frac{J^2(x \to +\infty; \mu)}{J^2(x \to -\infty; \mu)} \right|^2. \tag{36}$$

In (33), deforming the contour, for $x > 0$, to $\int_0^{-\infty}$ yields the relation

$$J(x; \mu) = \mathrm{e}^{-\pi\mu} J(-x; \mu)\,. \tag{37}$$

Using these relations in (32) and (36), we obtain

$$T = \mathrm{e}^{-2\pi\mu} \tag{38}$$

and

$$R = \mathrm{e}^{-2\Gamma}(1 - \mathrm{e}^{-2\pi\mu})^2 = \mathrm{e}^{-2\Gamma}(1 - T)^2\,, \tag{39}$$

in agreement with our heuristically derived results.

We now turn from the general dissipative Budden problem to our particular plasma-physics application, the conversion of a *magnetosonic* wave a to an *ion-hybrid* wave b in a two-ion-species (DT) tokamak plasma [3]. Under fusion conditions, the energetic alpha-particles gyroresonantly damp the hybrid wave [8].

We use the standard slab model, with nonuniform magnetic field $\boldsymbol{B} = \hat{\boldsymbol{z}}[1-(x/L_{\mathrm{B}})]B_0$. Because typically k_z and k_y are boundary-determined and small compared to k_x, which is dispersion-relation determined, we neglect them in D' and take $\boldsymbol{k} = \hat{\boldsymbol{x}}k_x$. We have the wave polarizations $\hat{\boldsymbol{e}}_a = \hat{\boldsymbol{e}}_{\mathrm{R}} \equiv (\hat{\boldsymbol{x}} + \mathrm{i}\hat{\boldsymbol{y}})/\sqrt{2}$ (right-circular) and $\hat{\boldsymbol{e}}_b = \hat{\boldsymbol{k}} = \hat{\boldsymbol{x}}$ (longitudinal); then (6) yields the two dispersion functions

$$D_a(k_x) = \epsilon_{\mathrm{R}} - k_x^2c^2/2\omega^2\,, \tag{40}$$

$$D_b(x) = \epsilon_{\perp}(x)\,, \tag{41}$$

in the cold-plasma model, where

$$\epsilon_{\perp}(x) \equiv \sum_i \frac{\omega_i^2}{\Omega_i^2(x)-\omega^2} \tag{42}$$

and

$$\epsilon_{\mathrm{R}} \equiv \sum_i \frac{\omega_i^2}{\Omega_i(\Omega_i+\omega)}\,. \tag{43}$$

(The x-dependence of ϵ_{R} is suppressed, since it varies slowly because it has no resonant denominator.) The magnetosonic dispersion function (40) has the form (9), with $\kappa_a = c^2/2\omega^2$ and $k_a^2 = \epsilon_{\mathrm{R}}/\kappa_a \equiv \omega^2/c_{\mathrm{A}}^2$ (defining the Alfven speed c_{A}). Setting $D_b = 0$, we obtain the x-dependent ion-hybrid frequency

$$\omega_H(x) = \frac{\omega_T^2\Omega_D^2(x) + \omega_D^2\Omega_T^2(x)}{\omega_T^2+\omega_D^2}\,, \tag{44}$$

which is then inverted to obtain the location $x_b(\omega)$ of this wave. Taylor-expanding $\epsilon_{\perp}(x)$ about x_b, we obtain (10), with

$$\kappa_b = \frac{\partial\epsilon_{\perp}}{\partial x} = \frac{2}{L_{\mathrm{B}}}\,\frac{\omega_H^2}{(\Omega_D^2-\Omega_T^2)^2}\,\frac{(\omega_T^2+\omega_D^2)^3}{\omega_T^2\omega_D^2}\,. \tag{45}$$

The coupling of the two waves is obtained from (7): $\eta = \epsilon_{\mathrm{R}}/\sqrt{2}$. Substituting these expressions into (19), and choosing equal densities for D, T ($n_D = n_T = \frac{1}{2}n_e$), we obtain the transmission coefficient

$$T = \exp[-(k_aL_{\mathrm{B}})/27]\,. \tag{46}$$

We now consider the effects of damping. The gyroresonant susceptibility of the alphas is [9,10]

$$\mathsf{D}'' = \frac{4\pi}{\omega^2}\int \mathrm{d}^3p\;\pi\delta(\omega-\Omega_\alpha(x)-k_zv_z)\,\boldsymbol{j}_1\boldsymbol{j}_1^* \times\left(-\frac{\partial}{\partial p_{\mathrm{g}}} - k_z\frac{\partial}{\partial p_z}\right)f_\alpha(p_{\mathrm{g}},p_z)\,, \tag{47}$$

where f_α is their Vlasov density (ignoring the contribution of spatial gradient), p_{g} and p_z are gyromomentum and parallel momentum, and

$$\boldsymbol{j}_1 = (e_\alpha\Omega_\alpha/k_x)(\hat{\boldsymbol{x}}J_1 - \mathrm{i}\hat{\boldsymbol{y}}\lambda J_1')\,, \tag{48}$$

with $\lambda \equiv k_xr_{\mathrm{g}}(p_{\mathrm{g}})$ the argument of the Bessel function J_1. We choose the α-distribution to be isotropic at their production energy $\mathcal{E}_\alpha = \frac{1}{2}m_\alpha v_\alpha^2$ (representing neonatal α's),

$$f_\alpha = \frac{n_\alpha\Omega_\alpha}{4\pi m_\alpha v_\alpha}\,\delta\!\left(p_{\mathrm{g}}\Omega_\alpha + \frac{p_z^2}{2m_\alpha} - \mathcal{E}_\alpha\right). \tag{49}$$

On using (49) in (47), we obtain

$$\mathsf{D}'' = \tfrac{1}{2}\pi\,\frac{\omega_\alpha^2}{\omega|k_z|v_\alpha} \times\left[\frac{1}{\lambda}\,\frac{\mathrm{d}}{\mathrm{d}\lambda}\{(\hat{\boldsymbol{x}}J_1 - \mathrm{i}\hat{\boldsymbol{y}}\lambda J_1')(\hat{\boldsymbol{x}}J_1 + \mathrm{i}\hat{\boldsymbol{y}}\lambda J_1')\}\right], \tag{50}$$

with λ evaluated at

$$p_{\mathrm{g}} = \left[\mathcal{E}_\alpha - \tfrac{1}{2}m_\alpha\left(\frac{\omega-\Omega_\alpha(x)}{k_z}\right)^2\right]\Big/\Omega_\alpha\,, \tag{51}$$

corresponding to local gyroresonance. For typical applications, $\lambda \sim v_\alpha/c_{\mathrm{A}} \sim \mathcal{O}(1)$, so we can approximate $J_1(\lambda)$ by its leading term $\frac{1}{2}\lambda$. Then (50) simplifies tremendously to

$$\mathsf{D}'' = \tfrac{1}{2}\pi\,\frac{\omega_\alpha^2}{\omega|k_z|v_\alpha}\hat{\boldsymbol{e}}_L\hat{\boldsymbol{e}}_L^* \tag{52}$$

within the gyroresonance layer $|\omega-\Omega_\alpha(x)| < |k_z|v_\alpha$, and zero outside it. Thus, in this approximation, the alphas gyroresonate only with the left-circular component ($\hat{\boldsymbol{e}}_L \equiv (\hat{\boldsymbol{x}} - \mathrm{i}\hat{\boldsymbol{y}})/\sqrt{2}$) of a wave, and indeed $\mathsf{D}''\cdot\hat{\boldsymbol{e}}_a = 0$, while

$$D_b'' \equiv \hat{\boldsymbol{x}}\cdot\mathsf{D}''\cdot\hat{\boldsymbol{x}} = \tfrac{1}{4}\pi\,\frac{\omega_\alpha^2}{\omega|k_z|v_\alpha}\,. \tag{53}$$

From (23) and (45), we obtain ν, and from (26), the damping decrement Γ:

$$\Gamma = \frac{2\sqrt{2}}{9}\left(\frac{n_\alpha}{n_e}\right)\left|\frac{k_a}{k_z}\right|\left(\frac{c_A}{v_\alpha}\right)(k_a L_B)\ . \tag{54}$$

This is composed of the small factor $n_\alpha/n_e \sim 10^{-3}$, the large factors $|k_a/k_z| \sim 10$ and $k_a L_B \sim 200$, and the order-one factor c_A/v_α. Thus Γ may be of order one, and offers a possible diagnostic test: the logarithm of the reflection coefficient should be linear in the alpha-density, by (36) and (54).

In summary, we have shown that the Budden equation is equivalent to a double conversion, and have calculated the effect of secondary-wave damping on primary-wave reflection, suggesting a diagnostic test. Extensions of this work should include:

(i) The effects of primary wave reflection from the plasma boundaries [11];

(ii) Effects of higher dimensionality, replacing the slab model of a tokamak by a more realistic poloidal section [3].

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