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Ray-based methods in multidimensional linear wave conversion[a]

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A tutorial introduction to the topic of linear wave conversion in multiple spatial dimensions is provided. The emphasis is on physical concepts, particularly those features of multidimensional conversion that are new and different from the more familiar ''mode conversion'' problem in one spatial dimension. After introductory comments, a brief review of WKB theory for vector wave equations in the absence of conversion is provided in order to introduce notation, terminology, and geometrical ideas. A primary theme of the discussion is that, although WKB (ray-based) methods break down in conversion regions, the ray geometry in the conversion region can be used to develop *local* wave equations that govern the two coupled wave channels undergoing conversion. These methods can be incorporated into ray-tracing algorithms providing, for the first time, the ability to follow the ''ray splitting'' associated with linear conversion in multidimensions, including the amplitude and phase changes associated with the conversion. © *2003 American Institute of Physics.* [DOI: 10.1063/1.1543579]

I. INTRODUCTION

This paper provides a tutorial introduction to recent results that were motivated by the need to incorporate linear conversion in multidimensional ray tracing algorithms, as discussed in Ref. 1. Prior ray-tracing algorithms could not treat conversion because of the associated ''ray splitting,'' where an incoming ray connects to *two* outgoing rays, a *transmitted* ray, and a *converted* ray. Linear conversion phenomena are exploited, for example, in heating strategies for tokamak fusion reactors as in Ref. 2.

In situations where the plasma geometry might be complicated (as in a tokamak or stellarator) and complex physical effects need to be included (e.g., flows, kinetic effects) it will often not be *a priori* obvious where conversions occur. Hence, any ray-based method must be able to (1) *discover* that the ray is entering a conversion region; (2) find the outgoing (transmitted and converted) rays; (3) find polarizations for the two waves undergoing conversion, allowing reduction to a local 2×2 wave equation; (4) evaluate the effective coupling coefficient (giving the transmission and conversion coefficients); and (5) fit the incoming WKB waves to outgoing WKB waves so that routine ray tracing methods can take over again.

Even when full-wave simulation codes are available (see, e.g., Refs. 3–6), ray tracing can provide much needed physical insight. Also, a fuller range of parameter studies should be possible with ray-based methods than with full-wave simulations, because the ray equations are systems of ordinary differential equations (ODEs), while the full-wave simulations involve systems of coupled partial differential equations (PDEs), or integral equations. In addition to its importance for applications, multidimensional conversion is interesting in its own right; a major theme of the present paper is that multidimensional conversion is far richer than its counterpart in one spatial dimension, and there is still much to learn.

Consider a general multicomponent linear wave equation governing electric fields in a weakly non-uniform plasma in two spatial dimensions $\mathbf{x}=(x_1,x_2)$,

$$\int d^2x'\,dt'\,\mathbf{D}(\mathbf{x},\mathbf{x}',t,t')\cdot\mathbf{E}(\mathbf{x}',t')=0,$$

or, written in component form:

$$\sum_{n=1}^{3}\int d^2x'\,dt'\,D_{mn}(\mathbf{x},\mathbf{x}',t,t')E_n(\mathbf{x}',t')=0,\quad m=1,2,3. \tag{1}$$

The goal is to find the electric field $\mathbf{E}(\mathbf{x},t)$ throughout a given spatial region (e.g., the interior of a tokamak) with fitting to appropriate boundary conditions. (It is important to note that the restriction here to two spatial dimensions and a

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three-component electric field are purely for the purposes of keeping the tutorial as concrete as possible. The methods described here can be extended to higher dimensions, and to wave equations with more components. In the general *N*-component problem, the components of the field might represent electric and magnetic fields, and fluid flow velocities, or in another context they might be the components of a spinor field.)

Here, and in the rest of the paper, we will assume that the plasma background properties are time-stationary, implying $D_{mn}(\mathbf{x},\mathbf{x}',t,t')=D_{mn}(\mathbf{x},\mathbf{x}',t-t')$. Our brief discussion of multidimensional WKB analysis of vector wave equations (in the absence of conversion) is based upon Ref. 7, where the extension to nonstationary media is also discussed.

We also assume that the wave propagation is nondissipative. Hence, the kernel is a 3×3 matrix-valued function of its arguments satisfying $D_{mn}(\mathbf{x},\mathbf{x}',t-t')=D^{*}_{nm}(\mathbf{x}',\mathbf{x},t'-t)$. Therefore, Eq. (1) can be derived from a variational principle,

$$\mathcal{A}\equiv\int d^2x\,d^2x'\,dt\,dt'\,\mathbf{E}^{\dagger}(\mathbf{x},t)\cdot\mathbf{D}(\mathbf{x},\mathbf{x}',t-t')\cdot\mathbf{E}(\mathbf{x}',t'). \tag{2}$$

This action principle is extremely useful for deriving conservation laws as well as the local wave equations appropriate to conversion regions, as shown in Ref. 8 (see Sec. III C).

Time stationarity allows us to write any solution of (1) as a linear superposition (in ω) of oscillations of the form $\mathbf{E}(x,t)=e^{-i\omega t}\tilde{\mathbf{E}}(x)$. We seek solutions of (1) that look, locally in $\mathbf{x}$, like plane waves of fixed frequency. That is, they are vector-valued eikonal waves

$$\mathbf{E}(\mathbf{x},t)=\psi(\mathbf{x})e^{i\theta(\mathbf{x})-i\omega t}\hat{\mathbf{e}}(\mathbf{x}). \tag{3}$$

It is implicit in all that follows that the physical field is the real part of (3). The *local wave vector* is defined as $\mathbf{k}(\mathbf{x})\equiv\nabla\theta(\mathbf{x})$. The function $\psi(\mathbf{x})$ is a slowly varying amplitude (assumed to be real and positive). The polarization $\hat{\mathbf{e}}(\mathbf{x})$ is a (complex) column vector with unit normalization; $\hat{\mathbf{e}}^{\dagger}(\mathbf{x})\cdot\hat{\mathbf{e}}(\mathbf{x})=1$, thus, $(\mathbf{E}^{\dagger}\cdot\mathbf{E})(\mathbf{x})\equiv|\mathbf{E}|^2(\mathbf{x})=\psi^2(\mathbf{x})$. The polarization is also assumed to be slowly varying. Eikonal solutions of the form (3) play the same fundamental role in the theory of waves in nonuniform plasmas as plane waves do in the uniform plasma theory. A general solution of the wave Eq. (1) would be a superposition of such eikonal waves with the amplitude and phase chosen so as to match boundary conditions. Motivation for the eikonal ansatz is provided in Sec. II. In this paper, we use the terms "eikonal" and "WKB" interchangeably.

Although reasonable on physical grounds, it is not *a priori* obvious that such eikonal solutions exist. In fact, WKB-type solutions typically do *not* exist globally. However, they often exist locally and, provided proper matching algorithms can be developed which "bridge the gaps" through the breakdown regions, they can be used to construct global solutions, using the multidimensional version of matched asymptotic expansions (often called *Maslov theory*) and described in Refs. 9–12. Breakdown of the WKB approximation may be caused by caustics, local absorption (e.g., due to gyroresonance), and linear conversion. The interested reader is referred to Ref. 13 for a discussion of caustics, and to Ref. 14 for a discussion of multidimensional gyroresonance and WKB matching conditions in the presence of kinetic effects.

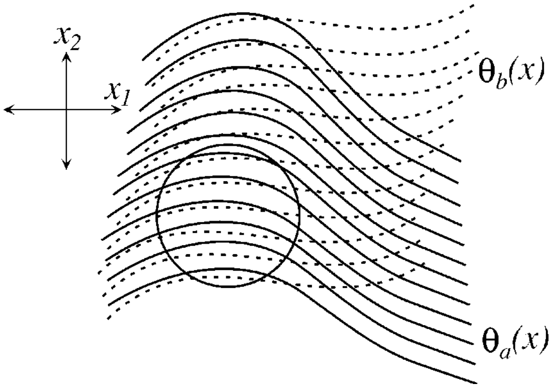

FIG. 1. A schematic showing the two phase functions θ_a and θ_b on the (x_1,x_2) plane. Linear conversion can occur only where the phase fronts are locally parallel and have similar spacings (implying that the local wave vectors $\mathbf{k}_a$ and $\mathbf{k}_b$ have nearly the same direction and magnitude). This condition is satisfied only in the region indicated by the circle. This is the spatial extent of the "conversion region." Note that the two group velocities of these two waves (not shown) can point in any direction.

Linear conversion occurs when two different wave types, with distinct polarizations and dispersion characteristics, *locally* near $\mathbf{x}_*=(x_{1*},x_{2*})$ have nearly equal *wave vectors* $\mathbf{k}_*=(k_{1*},k_{2*})$ for a given frequency ω. This local resonance results in the exchange of energy and action among the nearly degenerate waves.

The near-degeneracy condition has a simple geometrical interpretation within the WKB framework (see Fig. 1). Perform the following thought experiment: imagine there are two distinct eikonal solutions of the form (3) which we denote as a and b. These two wave fields are assumed to have distinct polarizations and dispersion characteristics (almost everywhere). Suppose a near-degeneracy $\nabla\theta_a\approx\nabla\theta_b$ occurs in the vicinity of the spatial point $\mathbf{x}_*$, and consider the local level sets at fixed time $t=t_0$ of the phases $\varphi_a(\mathbf{x},t)\equiv\theta_a(\mathbf{x})-\omega t$ and $\varphi_b(\mathbf{x},t)\equiv\theta_b(\mathbf{x})-\omega t$ [noting that $\mathbf{k}_{a*}\equiv\nabla\theta_a(\mathbf{x}_*)$ and $\mathbf{k}_{b*}\equiv\nabla\theta_b(\mathbf{x}_*)$]. The equality of gradients implies that the level sets of $\theta_a(\mathbf{x})$ and $\theta_b(\mathbf{x})$ are in close alignment and are not only parallel to one another but equally spaced. As time evolves, since the frequencies are equal for both waves, this relative phase relationship is preserved as the local crests and troughs propagate. If we now turn on the coupling, we can imagine wave a acting as a resonant drive for wave b (and *vice versa*).

It is important to note that it is a resonance of the local *phase* dynamics for the two waves that determines whether conversion occurs, not equality of the *group* velocities or the polarizations. When the group velocities are linearly independent, as they generically are, the conversion process cannot (even locally) be reduced to a one-dimensional problem in *x*-space. When the polarizations of the two waves are distinct, then it is possible to reduce (1) locally to a 2×2 version for the two resonant waves undergoing conversion, as will be shown in Sec. III C. (Note that in the special case where the two polarizations are identical, then the problem is locally reducible to a *scalar* wave equation and is a multidimensional variant of a *Landau–Zener crossing*.) See Ref. 15 for citations to the original papers and also the more recent discussions in Refs. 16–18.

Multicomponent wave equations of the form (1), and therefore the possibility for linear wave conversion, are ubiquitous throughout physics. Linear conversion occurs, for example, not only in the study of rf heating in fusion plasmas already mentioned, but also in ionospheric physics (see Ref. 19), as well as geophysics (see Ref. 8), atomic, molecular, and optical physics (see Ref. 16), and neutrino physics (see Ref. 20).

While there is a large physics literature on conversion in one dimension (see Refs. 15, 21–25, and references therein), and WKB-type methods were applied to vector wave equations as early as Rayleigh, to our knowledge there has been relatively little attention given to multidimensional linear conversion in vector wave equations. Examples of the latter in the physics literature are the papers of Bernstein and Friedland (see Ref. 26), the series of papers by Friedland and Kaufman (see Refs. 27–29), Littlejohn (see Ref. 18), Littlejohn and Flynn (see Ref. 30), Tracy and Kaufman (see Refs. 31,32), and Krasniak and Tracy (see Ref. 33), and in the mathematics literature those of Braam and Duistermaat (see Refs. 34,35) and de Verdiére (see Ref. 36).

This paper is organized as follows. In Sec. II we briefly summarize multidimensional WKB theory for vector wave equations (without conversion), keeping the medium time-stationary for the sake of simplicity. This is a very brief synopsis of the discussion in Ref. 7, and serves to introduce the phase space viewpoint we need for the following section. The primary theme of this section is that in multidimensions one must follow *families* of rays, and not single rays as in the one-dimensional theory.

We then discuss the ray-based treatment of linear conversion in Sec. III. Here it is important to note that, while the multidimensional WKB method requires that we follow families of rays, within the conversion region we can treat each incoming ray *separately*, with the final construction of the outgoing WKB fields obtained by superposing the results of each ray-by-ray conversion. This insight provides a great simplification and was first noted in Refs. 27–29. Results are quoted without proof. References are given to the literature where the result has already appeared, or motivated physically for those results that have not yet appeared in print. We end with a brief summary and discussion of future work.

II. MULTIDIMENSIONAL WKB THEORY FOR VECTOR WAVE EQUATIONS

First consider the simpler case of a uniform plasma. When the plasma is spatially uniform and time-stationary, the kernel $\mathbf{D}(\mathbf{x},\mathbf{x}',t,t')$ depends only upon $\mathbf{x}-\mathbf{x}'$ and $t-t'$. In this situation the system can be (formally) solved by Fourier methods. The convolution theorem implies that, in this case,

$$\tilde{\mathbf{D}}(\mathbf{k};\omega)\cdot\tilde{\mathbf{E}}(\mathbf{k};\omega)=0, \tag{4}$$

where $\tilde{\mathbf{D}}(\mathbf{k};\omega)$ (the *dispersion tensor*) and $\tilde{\mathbf{E}}(\mathbf{k};\omega)$ are the Fourier transforms of **D** and **E**, respectively. Nontrivial solutions of (4) exist only when $D(\mathbf{k},\omega)\equiv\det(\tilde{\mathbf{D}}(\mathbf{k},\omega))=0$, implying one scalar condition among the three variables (k_1,k_2,ω).

In the nonuniform case the Fourier convolution theorem does not apply. However, if the background plasma properties change smoothly over long length scales, then we can construct a local version of the dispersion tensor, denoted $\mathbf{D}(\mathbf{x},\mathbf{k},\omega)$. This is done using methods described fully in Refs. 7, 31, 9. Given the dispersion tensor, Eq. (1) can then be replaced by an equation of the form

$$\mathbf{D}(\mathbf{x},-i\nabla,i\partial_t)\cdot\mathbf{E}(\mathbf{x},t)=0. \tag{5}$$

The self-adjointness is preserved, hence this system also has an action principle,

$$\mathcal{A}'\equiv\int d^2x\,dt\,\mathbf{E}^\dagger(\mathbf{x},t)\cdot\mathbf{D}(\mathbf{x},-i\nabla,i\partial_t)\cdot\mathbf{E}(\mathbf{x},t). \tag{6}$$

The t-integration is over the finite interval $t_0\leqslant t\leqslant t_1$.

Eikonal solutions in the absence of conversion

Insert an ansatz of the form (3) into (6). For fixed ω, this ansatz has three unknown functions: $\theta(\mathbf{x})$, $\psi(\mathbf{x})$, and $\hat{\mathbf{e}}(\mathbf{x})$. Assume that the gradient and time derivative act primarily on the rapidly varying phase $\varphi(\mathbf{x},t)=\theta-\omega t$, to find (at leading order)

$$\mathcal{A}'\equiv\int d^2x\,dt\,\psi^2(\mathbf{x})\hat{\mathbf{e}}^\dagger(\mathbf{x})\cdot\mathbf{D}(\mathbf{x},\nabla\theta,\omega)\cdot\hat{\mathbf{e}}(\mathbf{x}). \tag{7}$$

Notice that the integrand is independent of time, hence the t-integration from t_0 to t_1 is trivial. The variation with respect to $\hat{\mathbf{e}}^\dagger(\mathbf{x})$ gives

$$\psi^2(\mathbf{x})\mathbf{D}(\mathbf{x},\nabla\theta,\omega)\cdot\hat{\mathbf{e}}(\mathbf{x})=0. \tag{8}$$

To have a nontrivial solution ($\psi(\mathbf{x})\neq 0$), the polarization $\hat{\mathbf{e}}(\mathbf{x})$ at each point $\mathbf{x}$ must be an eigenvector of the Hermitian matrix $\mathbf{D}(\mathbf{x},\nabla\theta(\mathbf{x}),\omega)$ with zero eigenvalue. Taking the determinant and defining $D(\mathbf{x},\nabla\theta(\mathbf{x}),\omega)\equiv\det(\mathbf{D}(\mathbf{x},\nabla\theta(\mathbf{x}),\omega))$ we arrive at the condition

$$D(\mathbf{x},\nabla\theta(\mathbf{x}),\omega)=0. \tag{9}$$

For each fixed value of ω, we see that $\theta(\mathbf{x})$ must satisfy a nonlinear partial differential equation (called the *eikonal* equation). For any $\theta(\mathbf{x})$ satisfying (9), the matrix $\mathbf{D}(\mathbf{x},\mathbf{k}(\mathbf{x}),\omega)$ has a zero eigenvalue $D_a(\mathbf{x},\mathbf{k}(\mathbf{x}),\omega)=0$, and $\hat{\mathbf{e}}_a(\mathbf{x})$ is the associated polarization. We have assumed we are not in a region of degeneracy; thus there is locally only one zero eigenvalue.

Since the phase $\theta(\mathbf{x})$ appears only through its gradient in (7), we can shift θ by a constant without changing the value of the action. Therefore the action has a continuous symmetry and, by Noether's theorem, there is an associated conservation law, the *action conservation law*, which can be used to propagate the amplitude $\psi(\mathbf{x})$, as described in Ref. 7.

In general cases, the most effective approach for finding solutions of (9) is via *ray tracing* (also known as the *method of characteristics*). Ray tracing, by definition, takes place in *phase space*, formed by adjoining the two-dimensional

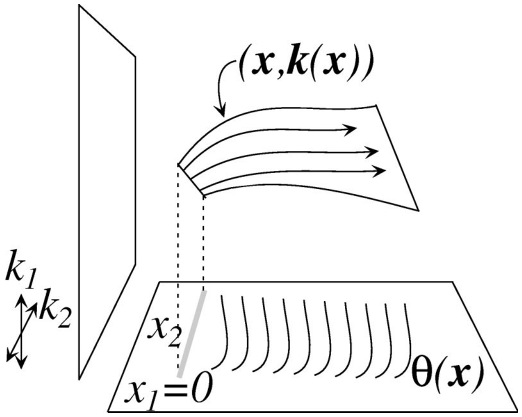

FIG. 2. Diagram showing how the phase function $\theta(\mathbf{x})$ locally defines a two-dimensional surface in the four-dimensional phase space. The boundary conditions for the phase are fixed, for example, on the one-dimensional curve $\mathbf{x}_B(x_2)=(0,x_2)$, indicated by the gray line in the (x_1,x_2)-plane. These boundary conditions are used to define a one-dimensional curve of ray initial conditions in the phase space. Under the evolution generated by Hamilton's equations, this family of rays sweeps out a two-dimensional surface.

spaces $\mathbf{x}$ and $\mathbf{k}$ into the four-dimensional space $\mathbf{z}=(\mathbf{x},\mathbf{k})$. Embracing the phase space viewpoint also leads to a significant conceptual clarification of the entire theory.

The notation $\mathbf{x}=(x_1,x_2)$ denotes a point in x-space, and $\mathbf{k}=(k_1,k_2)$ a wave vector (which plays the role of the conjugate momentum). (Here, $\mathbf{k}$ is an independent coordinate on the phase space and *not* the gradient of a particular phase function. The difference should be clear from context.) A point in the ray phase space will be denoted as $(\mathbf{x},\mathbf{k})=(x_1,x_2,k_1,k_2)$ or more compactly as $\mathbf{z}=(z_1,z_2,z_3,z_4)$. The latter notation emphasizes that in the Hamiltonian formulation the phase space variables appear on an equal footing.

We first consider the phase space interpretation of (9). Suppose $\theta(\mathbf{x})$ locally satisfies this nonlinear PDE. Noting that $\mathbf{k}(\mathbf{x})=\nabla\theta(\mathbf{x})$, we see that $(\mathbf{x},\mathbf{k}(\mathbf{x}))$ defines a two-dimensional surface in the four-dimensional phase space (see Fig. 2). This surface is called a *Lagrangian manifold*. In practice, the phase function $\theta(\mathbf{x})$ is constructed by launching a *family* of rays (one-dimensional curves) whose initial conditions are fixed at the boundary. This family of rays sweeps out the surface $(\mathbf{x},\mathbf{k})=(\mathbf{x},\nabla\theta(\mathbf{x}))$. Only by following such families of rays can a solution of (9) be constructed. Following single rays is not sufficient in multidimensional systems.

Returning to the dispersion tensor $\mathbf{D}(\mathbf{x},\mathbf{k})=\mathbf{D}(\mathbf{z})$ (from this point on we suppress the ω-dependence): at each point in the ray phase space the dispersion tensor is a Hermitian matrix with three real eigenvalues, $D_j(\mathbf{z})$, and three corresponding eigenvectors, $\hat{e}_j(\mathbf{z})$ with $j=a,b,c$. (We use Roman subscripts to denote *exact* eigenvalues and polarizations of the dispersion tensor in what follows.) Because the dispersion tensor is Hermitian, the eigenvectors form a complete orthonormal set for any $\mathbf{z}$: $\hat{\mathbf{e}}_m^\dagger(\mathbf{z})\cdot\hat{\mathbf{e}}_n(\mathbf{z})=\delta_{mn}$, $m,n=a,b,c$.

Consider one of the eigenvalues, for example $D_a(\mathbf{z})$. The locus of points for which $D_a(\mathbf{z})=0$ is the *dispersion surface* of wave a. We note that this surface also corresponds to a zero surface of $\det(\mathbf{D}(\mathbf{z}))\equiv D(\mathbf{z})=D_a(\mathbf{z})D_b(\mathbf{z})D_c(\mathbf{z})$, and that

$$\mathbf{D}(\mathbf{z})\cdot\hat{\mathbf{e}}_a(\mathbf{z})=0 \tag{10}$$

on this surface. Since a dispersion surface is defined by one scalar condition among the four variables (z_1,z_2,z_3,z_4), it is generically three-dimensional. On the dispersion surface of wave a, rays will propagate with the ray Hamiltonian $D(\mathbf{z})$. [N.B.: We can also use $D_a(\mathbf{z})$ as the ray Hamiltonian. This results in a reparametrization of the ray.]

Hamilton's equations (the *ray equations*) are most compactly written in terms of the Poisson bracket. For any two scalar functions, $f(\mathbf{z})$ and $g(\mathbf{z})$, the Poisson bracket is defined to be $\{f,g\}\equiv\nabla_x f\cdot\nabla_k g-\nabla_k f\cdot\nabla_x g$. The ray Hamiltonian $D(\mathbf{z})$ generates the ray evolution equations via

$$\dot{\mathbf{z}}\equiv\frac{d\mathbf{z}}{d\sigma}=\{D,\mathbf{z}\}. \tag{11}$$

A little algebra shows that these are $\dot{\mathbf{x}}=-\nabla_k D$ and $\dot{\mathbf{k}}=\nabla_x D$. The transformation between the use of the ray parameter σ and the physical time t is discussed in both Ref. 7 and Ref. 31. We note that any methodology proposed to compute the effect of linear conversion using ray methods must be *invariant* under reparametrization of the rays.

An arbitrary scalar function $f(\mathbf{z})$ will change following a ray via

$$\frac{df}{d\sigma}=\frac{d\mathbf{z}}{d\sigma}\cdot\nabla_z f|_{\mathbf{z}(\sigma)}=\{D,f\}. \tag{12}$$

An important consequence is that the Hamiltonian itself does not change following the ray: $\dot{D}=\{D,D\}=0$. Therefore, if we launch a ray on the dispersion surface of wave a, it will stay on this dispersion surface. Assume we are given initial conditions for a one-parameter family of rays that are consistent with the boundary conditions (see Ref. 7 for details). Now evolve each ray using Hamilton's equations (11). Note that in the four-dimensional phase space the rays satisfy uniqueness and, hence, *do not cross*.

The phase is constructed using $\theta(\mathbf{x})=\theta_0+\int_{\mathbf{x}_0}^{\mathbf{x}}\mathbf{k}(\mathbf{x})\cdot d\mathbf{x}$, where the integral is evaluated following rays launched from the boundary and θ_0 is the phase on the boundary. The amplitude ψ is propagated using the action conservation law mentioned earlier (see Ref. 7), and the polarization is propagated using methods described in Refs. 37 and 38.

As mentioned earlier, WKB methods break down for any of several reasons. When *caustics* occur, rays cross in x-space. This crossing leads to multiple phase values being assigned to the same point $\mathbf{x}$ and the prediction of singular behavior for the amplitude, an ambiguity that must be resolved by a more careful local treatment. (Although rays do not cross in the four-dimensional phase space, crossings can occur when the ray trajectories are projected onto the two-dimensional x-space.) There is by now a well-developed methodology for dealing with caustics (see Ref. 9). Another cause for the local failure of WKB is linear conversion, which is our main concern here.

III. RAY-BASED ANALYSIS OF MULTIDIMENSIONAL LINEAR CONVERSION

Away from conversion regions (and caustics), the amplitude and polarizations will vary slowly following a ray, justifying *a posteriori* the assumption that the derivatives in (5) act upon the phase to leading order. Within conversion regions, however, the polarization and amplitude vary rapidly, and the WKB ansatz is no longer valid.

We will restrict attention to the simplest type of conversion in multidimensions, where the rays are locally confined to a two-dimensional plane and exhibit an *avoided crossing* or *tunneling* region. More general types of behavior are possible (see Refs. 18, 34–36, 39), but these will be discussed elsewhere.

A. Detection of conversion following a ray

The basic idea used is due to Friedland and Goldner (see Ref. 40). Recall that the 3×3 dispersion tensor $\mathbf{D}(\mathbf{x},\mathbf{k})$ has three eigenvalues, $(D_a(\mathbf{z}),D_b(\mathbf{z}),D_c(\mathbf{z}))$. Assume we are following a ray on the dispersion surface $D_a(\mathbf{z})=0$. Conversion occurs when one of the other two eigenvalues becomes small. Any detection scheme must be invariant under reparametrization of the ray and *congruence transformations* (which shuffle the components of the vector field, and can thereby simplify the representation of the wave equation). Under a congruence transformation $\mathbf{Q}$ the electric field changes via $\mathbf{E}\equiv\mathbf{Q}\cdot\mathbf{E}'$, and the dispersion tensor via $\mathbf{D}\rightarrow\mathbf{D}'\equiv\mathbf{Q}^\dagger\cdot\mathbf{D}\cdot\mathbf{Q}$, leading to $\mathbf{D}'\cdot\mathbf{E}'=0$, with $\mathbf{D}'$ a Hermitian matrix. This ensures that there is still an action principle with $\mathcal{A}'=\int\mathbf{E}'^\dagger\cdot\mathbf{D}'\cdot\mathbf{E}'$.

General congruence transformations are nonlocal integral operators as discussed in Ref. 27. Here we restrict attention to congruence transformations given by 3×3 (invertible) matrices of complex constants. Sylvester's theorem states that the only congruence invariants are the *signs* of the eigenvalues, $\mathrm{sgn}(D_j)=\pm 1$, with zero eigenvalues unaffected. Therefore, the zero loci of the eigenvalues in phase space (the dispersion surfaces) will be unchanged, but away from the dispersion surfaces the values of $D_j(\mathbf{z})$ can be changed drastically.

Consider the effect of a congruence transformation upon the three quantities $\det(\mathbf{D})$, $\mathrm{Tr}(\mathbf{D})$, and what is often called the "second invariant" F (defined momentarily). The determinant is, of course, our ray Hamiltonian and is identically zero on the ray. If the dispersion tensor were only 2×2, the trace would be a good candidate for a "conversion detector." However, in the 3×3 case, following a ray on the dispersion surface $D_a=0$, we have $\mathrm{Tr}(\mathbf{D})=D_a+D_b+D_c=D_b+D_c$. Suppose D_b and D_c have opposite sign. Under congruence transformations $\mathrm{Tr}(\mathbf{D})$ could be positive for some choices of $\mathbf{Q}$ and negative for others. Therefore, the trace is *not* a good indicator for conversion in the 3×3 case. A better candidate is

$$\begin{aligned}F\equiv&(D_{22}D_{33}-|D_{23}|^2)+(D_{11}D_{33}-|D_{13}|^2)\\&+(D_{11}D_{22}-|D_{12}|^2)\\&=D_aD_b+D_aD_c+D_bD_c.\end{aligned}\tag{13}$$

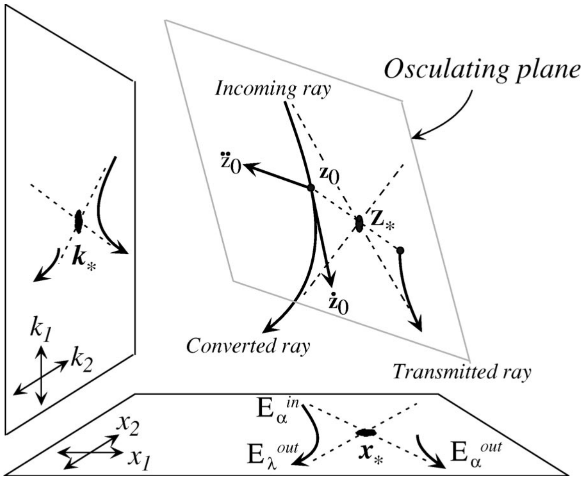

FIG. 3. Diagram of the geometry in the conversion region for a single ray of the incoming family. See text for details.

Following a ray of wave a we have $F(\sigma)=D_b(\sigma)D_c(\sigma)$. While the value of $F(\sigma)$ can change under congruence transformations, $\mathrm{sgn}(F)$ cannot change. Hence, a local minimum of $|F(\sigma)|$ is a good candidate for a "conversion detector."

B. Finding the outgoing rays

Once a local minimum of $|F(\sigma)|$ has been detected, we seek further evidence that conversion is, indeed, occurring. Denote the ray orbit parameter for the local minimum as σ_0. The simplest form of conversion in multidimensions has a local geometry like that in Fig. 3. That is, an "avoided crossing" with locally hyperbolic behavior confined to a two-dimensional plane embedded in the four-dimensional phase space. By analyzing the ray Hamiltonian locally in the vicinity of $\mathbf{z}(\sigma_0)$, it is possible to find the saddle point $\mathbf{Z}_*=(\mathbf{x}_*,\mathbf{k}_*)$ of the hyperbolas and, by searching along the line connecting $\mathbf{z}(\sigma_0)$ and $\mathbf{Z}_*$, find the second root of $\det(\mathbf{D})=0$ that lies nearby, thereby fixing the initial conditions for the transmitted ray.

Each incoming ray of the family smoothly connects onto a *converted* ray and *tunnels* through to a *transmitted* ray (see Fig. 4). Each ray-by-ray conversion looks much like an avoided crossing, or tunneling, phenomenon in one-dimensional systems, but since the plane can be oriented in any direction relative to *x*- or *k*-space, reduction to a one-dimensional problem in either *x*- or *k*-space is not possible.

We now consider how to reduce the 3×3 dispersion tensor to a local 2×2 form that will allow us to identify the local coupling constant, η. This reduction is necessary because, although it is possible to determine the magnitude of the coupling constant, and with it the transmission coefficient $\tau(\eta)=\exp(-\pi|\eta|^2)$, from the local ray geometry alone (see Ref. 1), the conversion coefficient, $\beta(\eta)$, requires both the amplitude and phase of η.

C. Reduction to the 2×2 form

Reduction to a 2×2 local wave equation in the vicinity of conversion can be carried out once we construct the polarizations of the "uncoupled" waves. (An alternative reduc-

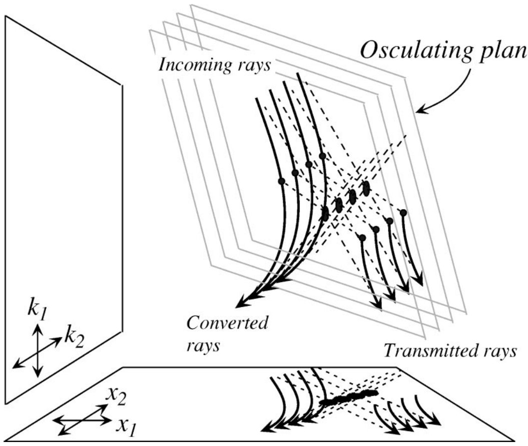

FIG. 4. The family of incoming rays converts ray-by-ray, forming the two outgoing families associated with the transmitted and converted waves. See text for details.

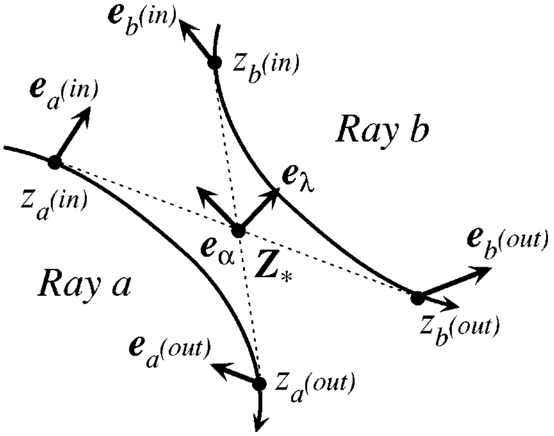

FIG. 5. A diagram of the local ray geometry and the polarizations in the osculating plane of the incoming ray near σ_0. The polarizations ($\hat{\mathbf{e}}_j^{(\mathrm{in})}$, $\hat{\mathbf{e}}_j^{(\mathrm{out})}$) with $j=a,b$ are slowly varying representative eigenvectors of the full 3×3 dispersion tensor outside (but close to) the conversion region. Their interpolation through the conversion region provides a smoothly varying polarization field whose values at $\mathbf{Z}_*$ we interpret as the polarizations of the two "uncoupled" waves.

tion method based directly upon the elimination of the nonresonant components of the multicomponent wave equation is discussed by Friedland and Goldner in Ref. 40 and by Friedland and Kaufman in Ref. 28.)

As mentioned earlier, away from a conversion region the polarization $\hat{\mathbf{e}}_a(\mathbf{z})$ is slowly varying following the ray. Suppose we have detected that another zero locus of the determinant lies nearby [say $D_b(\mathbf{z})=0$]. The following algorithm will allow us to locally replace the polarization fields $\hat{\mathbf{e}}_a(\mathbf{z})$, and $\hat{\mathbf{e}}_b(\mathbf{z})$, which are rapidly varying in the conversion region, with two *constant* polarizations $\hat{\mathbf{e}}_\alpha$ and $\hat{\mathbf{e}}_\lambda$, the polarizations of the "uncoupled" waves at the conversion point. Greek indices denote "uncoupled" quantities. Note that, although $\hat{\mathbf{e}}_\alpha$ and $\hat{\mathbf{e}}_\lambda$ are not eigenvectors of the original dispersion matrix $\mathbf{D}$, they will be eigenvectors of the reduced dispersion matrix *if the coupling is turned off*. Hence, they should be orthogonal to one another $\hat{\mathbf{e}}_\alpha^\dagger\cdot\hat{\mathbf{e}}_\lambda=0$, which provides another consistency check for the algorithm.

Referring to Fig. 5, we first find examples of the slowly-varying incoming and outgoing polarizations following the ray a (which transitions smoothly between the incoming ray of uncoupled type λ and the outgoing converted ray of type α). Do the same on the second ray (of type b). Now interpolate between these two pairs of polarizations to the saddle point $\mathbf{Z}_*$ to find $\hat{\mathbf{e}}_\alpha$ and $\hat{\mathbf{e}}_\lambda$.

Returning to the action principle (6), insert the ansatz

$$\begin{aligned}\mathbf{E}(\mathbf{x},t)&=e^{i\mathbf{k}_*\cdot(\mathbf{x}-\mathbf{x}_*)-i\omega t}\tilde{\mathbf{E}}(\mathbf{x})\\&\equiv e^{i\mathbf{k}_*\cdot(\mathbf{x}-\mathbf{x}_*)-i\omega t}[E_\alpha(\mathbf{x})\hat{\mathbf{e}}_\alpha+E_\lambda(\mathbf{x})\hat{\mathbf{e}}_\lambda].\end{aligned}\tag{14}$$

This ansatz reflects the fact that, near $\mathbf{x}_*$, waves in both channels look locally like plane waves with wave number $\mathbf{k}_*$. The *complex* scalar amplitude functions, E_α and E_λ, include all the effects of the coupling. These local amplitudes have rapid variation in both amplitude and phase in the conversion region, but connect smoothly onto the incoming and outgoing WKB wave fields.

Acting with the gradient on any function of the form $e^{i\mathbf{k}_*\cdot\mathbf{x}}\phi(\mathbf{x})$, we find $-i\nabla(e^{i\mathbf{k}_*\cdot\mathbf{x}}\phi(\mathbf{x}))=e^{i\mathbf{k}_*\cdot\mathbf{x}}(\mathbf{k}_*-i\nabla)\phi(\mathbf{x})$, hence the commutation of the gradient operator with multiplication by the carrier has the effect of shifting the origin in k-space to $\mathbf{k}_*$. (The origin in x-space, of course, can be shifted to $\mathbf{x}_*$ by simple translation.) By these means, we can shift the origin in phase space to the conversion point $\mathbf{Z}_*$ (see Ref. 9 for a more complete discussion). This result is clearly demonstrated in the variational principle. Inserting the ansatz (14) into the action gives (the t-integration from t_0 to t_1 is again trivial and the constant ignored)

$$\mathcal{A}'\equiv\int d^2x\,\tilde{\mathbf{E}}^\dagger(\mathbf{x})\cdot\mathbf{D}(\mathbf{x},\mathbf{k}_*-i\nabla)\cdot\tilde{\mathbf{E}}(\mathbf{x}).\tag{15}$$

The variation with respect to E_α and E_λ gives the reduced wave equation,

$$\begin{pmatrix}\hat{D}_{\alpha\alpha} & \hat{D}_{\alpha\lambda}\\ \hat{D}_{\alpha\lambda}^\dagger & \hat{D}_{\lambda\lambda}\end{pmatrix}\begin{pmatrix}E_\alpha(\mathbf{x})\\ E_\lambda(\mathbf{x})\end{pmatrix}=0,\tag{16}$$

with $\hat{D}_{jk}\equiv\hat{\mathbf{e}}_j^\dagger\cdot\mathbf{D}(\mathbf{x},\mathbf{k}_*-i\nabla)\cdot\hat{\mathbf{e}}_k$ and $j,k=(\alpha,\lambda)$. The operators $\hat{D}_{jk}$ are linear combinations of the original entries of $\mathbf{D}$ since the polarization vectors are constant. We wish to simplify this system of equations in the vicinity of the conversion point.

The 2×2 form of the wave operator has the following 2×2 *reduced dispersion matrix*:

$$\mathbf{D}(\mathbf{z})=\begin{pmatrix}D_{\alpha\alpha}(\mathbf{z}) & D_{\alpha\lambda}(\mathbf{z})\\ D_{\alpha\lambda}^*(\mathbf{z}) & D_{\lambda\lambda}(\mathbf{z})\end{pmatrix}\equiv\begin{pmatrix}D_\alpha(\mathbf{z}) & \tilde{\eta}(\mathbf{z})\\ \tilde{\eta}^*(\mathbf{z}) & D_\lambda(\mathbf{z})\end{pmatrix},\tag{17}$$

with $D_{jk}(\mathbf{z})\equiv\hat{\mathbf{e}}_j^\dagger\cdot\mathbf{D}(\mathbf{z})\cdot\hat{\mathbf{e}}_k$ and $j,k=(\alpha,\lambda)$. This dispersion matrix is expanded to leading order about the conversion point by first writing $\mathbf{z}=\mathbf{Z}_*+(\mathbf{z}-\mathbf{Z}_*)$ and keeping only terms linear in $(\mathbf{z}-\mathbf{Z}_*)$. The diagonal terms are zero at leading order, since the polarizations project $\mathbf{D}$ onto the *uncoupled* wave channels. The slowly-varying off-diagonal terms are constant at leading order. Therefore,

$$\mathbf{D}(\mathbf{z}) \approx \begin{pmatrix} (\mathbf{z}-\mathbf{Z}_*)\cdot\nabla_z D_\alpha & \tilde{\eta} \\ \tilde{\eta}^* & (\mathbf{z}-\mathbf{Z}_*)\cdot\nabla_z D_\lambda \end{pmatrix}. \tag{18}$$

The eight constants $\nabla_z D_\alpha$ and $\nabla_z D_\lambda$ are evaluated at the conversion point. This linearized dispersion matrix has a related operator which is obtained by associating $\mathbf{k}-\mathbf{k}_* \to -i\nabla$. Inserting this 2×2 operator into the variational principle and carrying out the variation leads to the 2×2 wave equation,

$$\begin{pmatrix} \alpha_x\cdot\mathbf{x}-i\alpha_k\cdot\nabla & \tilde{\eta} \\ \tilde{\eta}^* & \lambda_x\cdot\mathbf{x}-i\lambda_k\cdot\nabla \end{pmatrix} \begin{pmatrix} E_\alpha(\mathbf{x}) \\ E_\lambda(\mathbf{x}) \end{pmatrix} = 0. \tag{19}$$

Here the two-vectors α_x and α_k are defined as

$$\alpha_x \equiv \left(\frac{\partial D_\alpha}{\partial x_1}, \frac{\partial D_\alpha}{\partial x_2}\right), \quad \alpha_k \equiv \left(\frac{\partial D_\alpha}{\partial k_1}, \frac{\partial D_\alpha}{\partial k_2}\right) \tag{20}$$

(etc. for the vectors λ_x and λ_k) with the derivatives evaluated at the saddle point. The coupling constant is

$$\tilde{\eta} \equiv \hat{\mathbf{e}}_\alpha^\dagger \cdot \mathbf{D}(\mathbf{Z}_*) \cdot \hat{\mathbf{e}}_\lambda. \tag{21}$$

Notice that this definition of $\tilde{\eta}$ includes the phase, needed below.

D. Solving for the WKB connection coefficients

The field $\tilde{\mathbf{E}}(\mathbf{x})$ of (14) is most definitely *not* of WKB form. However, as we move away from the conversion region the scalar fields $E_\alpha(\mathbf{x})$ and $E_\lambda(\mathbf{x})$ [along with the plane wave part $\exp(i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}_*)-i\omega t)$] will match onto the incoming and outgoing WKB fields. Hence, we need to find connection coefficients that relate E_α and E_λ from the incoming to the outgoing regions and then fit these to incoming and outgoing WKB solutions.

While (19) is a great simplification over (5), we are still faced with solving a pair of coupled PDEs with nonconstant coefficients. Once again, the use of Hamiltonian methods provides a powerful tool for finding the solution. As shown in Ref. 31, by carrying out a linear canonical transformation ($\mathbf{z}\to\mathbf{z}'=\mathbf{M}\cdot\mathbf{z}$ with $\mathbf{M}$ a 4×4 symplectic matrix) it is possible to recast this local 2×2 problem into a much simpler form. The use of linear canonical transformations in phase space is related to transformations of the wave equation (19) that are generalizations of Fourier transformations, as discussed in Refs. 9 and 31. A proper choice of representation reduces the multidimensional PDEs to *first order* ODEs which are straightforward to solve.

Relating the incoming and outgoing field amplitudes gives the *transmission* and *conversion* coefficients (see Ref. 31),

$$\tau(\eta) = e^{-\pi|\eta|^2}, \quad \beta(\eta) \equiv \frac{(2\pi\tau)^{1/2}}{\eta\Gamma(-i|\eta|^2)}, \tag{22}$$

where the *normalized coupling constant* is defined as

$$\eta = \frac{\tilde{\eta}}{|\mathcal{B}|^{1/2}}, \tag{23}$$

with $\mathcal{B}\equiv\{D_\alpha, D_\lambda\}$ the Poisson bracket of the uncoupled dispersion functions. Therefore, the family of incoming rays, which carry with them phase and amplitude information, is sufficient for fixing the phase and amplitude of the two outgoing families of rays associated with the transmitted and converted waves. These two families of rays generate a line of saddle points in phase space which, when projected onto *x*-space, form a line there, too (see Fig. 4). Hence, in *x*-space conversion will occur along a one-dimensional curve whose position depends upon the entire incoming family of rays [i.e., the incoming phase function $\theta_a(\mathbf{x})$].

All that is lacking at this point is the local field structure in *x*-space in the conversion region itself $[E_\alpha(\mathbf{x})$ and $E_\lambda(\mathbf{x})]$. The local form of the fields can be found, and will be discussed in detail in a future paper.

IV. SUMMARY AND CONCLUSIONS

In this tutorial, we have presented an introduction to a recently developed ray-based approach to multidimensional linear conversion, appropriate for incorporation into ray tracing algorithms. Much of the conceptual difficulty associated with linear conversion in multidimensions stems from the fact that the underlying geometry is high-dimensional and, therefore, difficult to visualize. However, by following ray trajectories, it is possible to extract the necessary information for finding the WKB connection coefficients. Except for the details of the field structure within the conversion region itself, the information required for computing the transmission and conversion coefficients is embodied in the coupling constant η. Calculation of η does not require full solution of the problem, but only the ability to project the 3×3 dispersion tensor onto a local ''uncoupled'' basis.

There is still much to learn regarding this problem, and the application to realistic problems with complicated geometries has only begun (see Ref. 1). Our goal in this tutorial has been to provide the interested reader with a conceptual roadmap to pursue further explorations, and—hopefully—with a sense that there are interesting things left to discover.

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