Linear-conversion theory of energetic minority-ion Bernstein-wave propagation across gyroresonance in nonuniform magnetic field

A. J. Brizard[a)] and A. N. Kaufman
Lawrence Berkeley National Laboratory, University of California, Berkeley, California 94720

(Received 18 July 1995; accepted 9 October 1995)

An energetic minority-ion population, such as neonatal fusion alphas, can support a Bernstein wave whose frequency is a harmonic of their gyrofrequency. The sign of the wave energy depends on the local wave vector, whose rate of change is proportional to the magnetic-field gradient. As a result of the field nonuniformity, the wave crosses the gyroresonance layer and its energy flips sign. This results in energy transfer to a gyroballistic mode (which exists only in the resonance layer), with a conversion coefficient exactly equal to 2. © *1996 American Institute of Physics.*
[S1070-664X(96)04701-5]

I. INTRODUCTION

Recent interest in negative-energy Bernstein waves, supported by an *inverted* population of energetic minority ions (such as charged fusion products),[1] warrants the investigation of their properties. In this paper, we study their propagation in a *nonuniform* magnetic field, and show that wave refraction ($d\mathbf{k}/dt \equiv -\partial\omega/\partial\mathbf{x}$) causes the Bernstein-wave energy (which is $\mathbf{k}$ dependent) to change *sign* when the wave crosses the gyroresonance layer.

This situation presents us with the following paradox: How can a wave change the sign of its energy while maintaining overall energy conservation? We resolve this paradox by showing that the Bernstein wave changes its energy sign by transferring energy to a gyroballistic mode,[2] which exists only in the gyroresonance layer. (A ballistic mode represents a perturbed distribution with vanishing charge density and self-consistent potential.)

In order to obtain explicit analytical results, we make certain simplifying assumptions concerning the energetic minority-ion distribution. Neonatal charged fusion products (e.g., alphas) have a monoenergetic distribution:[3] $f_0(p_\parallel, p_g) \sim \delta(\mathscr{E} - \mathscr{E}_0)$, where $\mathscr{E} = p_\parallel^2/2m + p_g\Omega$ is the kinetic energy ($p_\parallel$ is the parallel momentum, $p_g = mc\mu/e$ is the gyromomentum, and $\Omega = eB/mc$ is the gyrofrequency), and $\mathscr{E}_0$ is the birth energy of the charged fusion products. Hence, for $p_g < p_0 \equiv \mathscr{E}_0/\Omega$, we have $\partial F_0/\partial p_g > 0$, where $F_0(p_g) \equiv \int dp_\parallel\, f_0(p_\parallel, p_g) \sim (p_0 - p_g)^{-1/2}$, i.e., the p_g distribution is inverted. In this paper, we choose the model distribution:[4] $F_0(p_g) \sim \delta(p_g - p_0)$, which significantly simplifies our analysis. In addition, we choose the wave vector $\mathbf{k} = \hat{x}k_x$, i.e., $k_z = 0$ (often used for Bernstein waves) and $k_y = 0$ (used here for simplicity; the finite-k_y problem is more complicated to solve analytically).

The remainder of this paper is organized as follows. First, in Sec. II, we show that a Bernstein wave supported by an inverted population of energetic minority ions can have negative energy. Moreover, in a nonuniform magnetic field, the wave energy is shown to change its sign repeatedly as the wave crosses the gyroresonance layer. (In Appendix A, the linearized Vlasov–Poisson equations for the Bernstein wave, near the lth harmonic of the minority-ion gyrofrequency, are derived in the guiding-center representation.[5]) Next, in Sec. III, we derive a set of coupled equations near the gyroresonance layer. Here, the two coupled modes are the energetic minority-ion Bernstein mode and the gyroballistic mode, which exists only inside the resonance layer. The gyroresonance crossings can be divided into two types. For crossings of the first type, the energy sign goes from negative to positive, while for crossings of the second type, the energy sign goes from positive to negative. The analysis for each crossing type is quite similar; the first type is analyzed in Sec. III and, in Appendix B, the results for the second type are summarized. In Sec. IV, we construct a variational principle for the coupled equations, which are shown to possess an exact invariant corresponding to total wave energy. Finally, in Sec. V, we study the mode conversion process that takes place when a Bernstein wave crosses the gyroresonance layer and changes the sign of its energy. By the conservation of energy, this implies that some energy is converted to the gyroballistic mode (inside the gyroresonance layer). For both types of gyroresonance crossing, we show that the energy conversion coefficient $C = 2$, while the transmission coefficient $T = -1$.

II. NEGATIVE-ENERGY ENERGETIC MINORITY-ION BERNSTEIN WAVE

We begin with a simple model for a magnetic-fusion plasma composed of a cold majority-ion (M) fluid, with density n_M, an electron fluid, and an energetic minority-ion (m) population described by a linear Vlasov equation (in the guiding-center representation). Next, we use a slab model for the background magnetic field $\mathbf{B}(x) \equiv \hat{z}(1 - x/L)B_0$, where $x = 0$ is the location of the lth-harmonic minority gyroresonance layer: $\omega = l\Omega_m(x=0)$ (with $l \sim 5$). (The use of the cold-fluid approximation for the majority species is justified for $l \geqslant 3$, since the thermal effect of the majority ions can be neglected compared to that of the energetic minority ions.)

We first derive the local dielectric function and the Bernstein-wave dispersion relation *away* from the resonance layer, where eikonal methods are valid. We treat the wave as longitudinal ($\mathbf{E} = -\nabla\phi$) with a definite (real) wave frequency ω, and take $\mathbf{k} = \hat{x}k_x$ (i.e., $k_y = 0 = k_z$). We express the linear

[a)]E-mail. brizard@tops.lbl.gov

Vlasov perturbation δf of the minority ions in terms of a generating function $S(p_g,\theta;p_z;X,t)$ (in the guiding-center representation):[6]

$$\delta f \equiv \{S,f_0\} = \frac{\partial S}{\partial \theta}\frac{\partial f_0}{\partial p_g}, \tag{1}$$

where the gyrophase θ and the gyromomentum $p_g = (mc/e)\mu$ are canonically conjugate, p_z is parallel momentum, X is the x component of guiding-center position, and $f_0(p_z,p_g,X)$ is the unperturbed minority-ion distribution. [Because $k_z=0$ and $k_y=0$, the terms in $\partial f_0/\partial p_z$ and $\partial f_0/\partial X$, respectively, do not appear in Eq. (1); see Appendix A for details.]

Within the eikonal approximation, we take

$$\begin{aligned} S(p_g,\theta,p_z;X,t) &= \tilde{S}_l(p_g,p_z;X)\exp[i\Theta(X)+il\theta-i\omega t], \\ \phi(x,t) &= \tilde{\phi}(x)\exp[i\Theta(x)-\omega t], \end{aligned} \tag{2}$$

where $\Theta(x)$ is the eikonal phase $[k_x(x)\equiv d\Theta/dx]$, while $\tilde{S}_l$ and $\tilde{\phi}$ are slowly varying functions of X and x, respectively. From Eqs. (A7e) and (A14) of Appendix A, respectively, the evolution equation for $\tilde{S}_l$ and the Poisson equation for $\tilde{\phi}$ are

$$-i[\omega - l\Omega_m(x)]\tilde{S}_l(p_g,p_z;x) = e_m J_l(k_x\rho)\tilde{\phi}(x), \tag{3}$$

$$\epsilon_M(\omega)k_x^2\tilde{\phi}(x) = 4\pi e_m \int J_l(k_x\rho)\,\delta\tilde{f}_l(p_g,p_z;x), \tag{4}$$

where $\delta\tilde{f}_l = il\tilde{S}_l\,\partial f_0/\partial p_g$, $\epsilon_M(\omega) = -\omega_M^2/(\omega^2-\Omega_M^2)$ is the majority dielectric function (in the cold-fluid approximation), the Bessel function J_l has the argument $k_x(x)\rho(p_g,x)$, with $\rho(p_g,x) = [2p_g/m\Omega_m(x)]^{1/2}$, and $\int = 2\pi m\Omega_m \int_0^\infty dp_g \times \int_{-\infty}^\infty dp_z$.

From Eq. (3), we note that $\tilde{S}_l$ is independent of p_z (because $k_z=0$), and thus in Eq. (4) we can immediately perform the p_z integration, defining $F_0(p_g) \equiv 2\pi m\Omega_m \int dp_z\, f_0(p_g,p_z)$; throughout the rest of this paper, we neglect the X dependence of f_0, since it does not play an important role. Eliminating $\tilde{S}_l$ by Eq. (3), we then obtain $\epsilon(k_x,x;\omega)\tilde{\phi}(x)=0$, where the *local* dielectric function is

$$\epsilon(k_x,x;\omega) = \epsilon_M(\omega) + \chi_l(k_x,x;\omega), \tag{5}$$

with the minority susceptibility:

$$\chi_l(k_x,x;\omega) = -\frac{\omega_m^2}{\Omega_m^2}\frac{\omega}{\omega - l\Omega_m(x)}\,\beta(k_x), \tag{6}$$

where $\omega_m = (4\pi n_m e_m^2/m)^{1/2}$ is the minority-ion plasma frequency ($\omega_m \ll \omega_M$), and

$$\beta(k_x) \equiv \langle \lambda^{-1}\, dJ_l^2/d\lambda \rangle, \tag{7}$$

with $\lambda \equiv k_x\rho(p_g)$ and

$$\langle \cdots \rangle \equiv \int dp_g(\cdots)F_0 \bigg/ \left(\int dp_g\, F_0\right), \tag{8}$$

is the average over the p_g distribution. We note that if $F_0(p_g)$ has a maximum away from $p_g=0$ (i.e., if there exists a region for $p_g>0$ where $\partial F_0/\partial p_g>0$), then $\beta(k_x)$ is an *oscillating* function of k_x and, more importantly, it can *change sign* by going through zero.

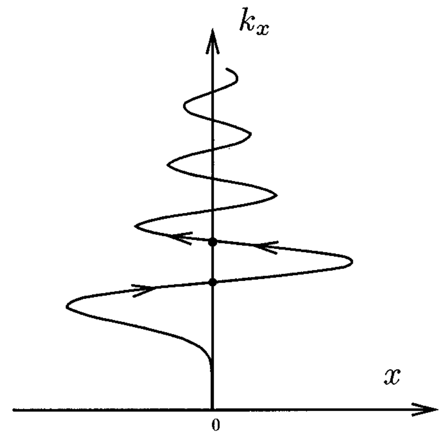

FIG. 1. Plot of the energetic minority-ion Bernstein ray orbit (10) in ray phase space (x,k_x). As the k_x coordinates of the ray increases ($k_x = \omega/L > 0$), the ray orbit eventually crosses the gyroresonance layer at $x=0$ (first ●), where the energy sign goes from positive to negative. As k_x increases further, the orbit crosses the layer again (second ●), where the energy sign goes from negative to positive. An infinite number of crossings occur, as shown in the figure, with the maxima for the x coordinate of the ray orbit decreasing as a function of k_x.

Solving $\epsilon(k_x,x;\omega)=0$ for ω, we obtain the *local* dispersion relation:

$$\omega(k_x,x) = l\Omega_m(x)[1-\alpha\beta(k_x)] = \omega\left(1 - \frac{x}{L} - \alpha\beta(k_x)\right), \tag{9}$$

where $\alpha \equiv (l^2 - \Omega_M^2/\Omega_m^2)\omega_m^2/\omega_M^2 \ll 1$ and $|\beta|<1$. Setting $\omega(k_x,x)=\omega$, we then obtain the ray orbit:

$$x(k_x)/L = -\alpha\beta(k_x), \tag{10}$$

where the ray velocity is given by the Hamilton equation $dk_x/dt = -\partial\omega/\partial x = \omega/L$. See Fig. 1 for this orbit, where we take $l=5$ and

$$F_0(p_g) \sim \delta(p_g - p_0), \tag{11}$$

so that (7) becomes

$$\beta(k_x) = 2\lambda^{-1}J_l(\lambda)J_l'(\lambda), \tag{12}$$

with $\lambda = k_x\rho(p_0)$. Note that, from Eq. (10), the ray crosses the gyroresonance layer (at $x=0$) whenever β vanishes; from Eq. (12), the ray crosses the layer repeatedly as λ alternately passes through the zeros of J_l and J_l'. The model distribution (11) mimics neonatal charged fusion products,[4] where p_0 is their gyromomentum at birth; it is chosen here for its analytic tractability. A more realistic distribution would show similar repeated crossings.[7]

The Bernstein-wave energy density $W \equiv \omega(\partial\epsilon/\partial\omega) \times k_x^2|\tilde{\phi}|^2/4\pi$ is obtained from Eqs. (5) and (9). We find

$$\omega\frac{\partial\epsilon}{\partial\omega} = \frac{|\epsilon_M|}{\alpha\beta(k_x)} = \left(\frac{\omega_M^2\Omega_m}{\omega_m(l^2\Omega_m^2 - \Omega_M^2)}\right)^2 \frac{1}{\beta(k_x)}, \tag{13}$$

and thus the wave energy W has the same sign as β. Referring to Fig. 1, we see that W changes sign (as β goes through zero) whenever the ray crosses the gyroresonance layer at $x=0$. As $x\to 0$, $\partial\epsilon/\partial\omega \to \infty$, and so (within the eikonal approximation), the potential amplitude $\tilde{\phi}\to 0$. Of course, in this singular (resonance) region, a noneikonal treatment is required, which we study in the next section.

To interpret the negative wave energy, we consider the change in the gyromomentum of an ion to second order in S. Using Lie-transform theory,[8] a change Δp_g in gyromomentum can be *generated* by the (real) function S_r:

$$\Delta p_g \equiv [\exp(\{S_r,\cdot\})-1]p_g$$
$$=\{S_r,p_g\}+\tfrac{1}{2}\{S_r,\{S_r,p_g\}\}+\mathcal{O}(S_r^3), \tag{14}$$

where $S_r\equiv S+S^*$ is written in terms of a complex function $S(p_g,\theta;X,t)\equiv S_l(p_g,X)e^{i(l\theta-\omega t)}$. The first-order term in Eq. (14), $\{S_r,p_g\}=(ilS-ilS^*)$, vanishes upon averaging over θ or time. The change in gyromomentum is nonvanishing at second order:

$$\Delta p_g=\tfrac{1}{2}\{S,\{S^*,p_g\}\}+\text{c.c.}, \tag{15}$$

which becomes, after averaging over θ or time,

$$\Delta p_g=\frac{1}{2}\,l^2\,\frac{\partial|S_l|^2}{\partial p_g}. \tag{16}$$

Hence, the kinetic energy per particle changes by $\Omega_m\,\Delta p_g$, and the wave energy density is $W=n_m\Omega_m\,\Delta p_g$. Using (16) and (3) for S_l, we obtain agreement with (13). Thus, the negativity of W comes from the p_g dependence of S_l in (16).

In this section, we have shown that a Bernstein wave supported by an inverted energetic minority-ion population changes the sign of its energy in a nonuniform magnetic field when the wave crosses the gyroresonance layer. For the model distribution (11), these crossings are characterized by the energy sign going from positive to negative (corresponding to the zeros of J_l) or going from negative to positive (corresponding to the zeros of J_l'). In the next section, we derive a set of coupled equations corresponding to the coupling of two modes near the gyroresonance layer.

III. COUPLED WAVES NEAR THE GYRORESONANCE LAYER

Near the gyroresonance layer, the eikonal approximation (2) is invalid. The coupled equations (3)–(4), however, still have the same *form*:

$$-i[\omega-l\Omega_m(x)]S_l(p_g;x)=e_mJ_l(\hat{k}_x\rho)\,\phi(x), \tag{17}$$

$$\epsilon_M(\omega)\hat{k}_x^2\phi(x)=4\pi e_m\int dp_g\;J_l(\hat{k}_x\rho)\times ilS_l(p_g;x)F_0'(p_g), \tag{18}$$

but now $\hat{k}_x$ is the operator $-i\partial/\partial x$, whereas $S_l(p_g;x)$ and $\phi(x)$ are (noneikonal) functions of x. By substituting $l\Omega_m(x)\equiv\omega(1-x/L)$ into $[\omega-l\Omega_m(x)]$, the left side of Eq. (17) becomes $-i(\omega x/L)S_l$. Because of the singularity at $x=0$, it is better to Fourier transform to the k_x representation, wherein x becomes the operator $i\partial/\partial k_x$ and $-i(\omega x/L)\times S_l(p_g;x)\to(\omega/L)\partial S_l(p_g;k_x)/\partial k_x$. The coupled equations (17)–(18) now read as

$$\left(\frac{\omega}{L}\right)\frac{\partial S_l}{\partial k_x}\,(p_g;k_x)=e_mJ_l(\lambda)\,\phi(k_x), \tag{19}$$

$$\epsilon_M k_x^2\phi(k_x)=-4\pi ile_mn_m\left\langle\frac{\partial}{\partial p_g}\,(J_lS_l)\right\rangle(k_x), \tag{20}$$

where integration by parts over p_g was performed [the boundary term at $p_g=0$ vanishes since $J_l(0)=0$ for $l\neq 0$], and $\langle\cdots\rangle$ is defined in Eq. (8).

In what follows, it is convenient to introduce the function $R_l(p_g;k_x)\equiv 2p_g\,\partial S_l(p_g;k_x)/\partial p_g$, so that, on the right side of Eq. (20), the identity $\partial(J_lS_l)/\partial p_g=(\lambda J_l'S_l+J_lR_l)/2p_g$ can be used (note that $\partial\lambda/\partial p_g=\lambda/2p_g$). The differential equation satisfied by $R_l(p_g;k_x)$,

$$\left(\frac{\omega}{L}\right)\frac{\partial R_l(p_g;k_x)}{\partial k_x}=\lambda J_l'(\lambda)e_m\phi(k_x), \tag{21}$$

is obtained by operating on Eq. (19) with $2p_g\,\partial/\partial p_g$. The Poisson equation (20) can thus be rewritten as

$$e_m\phi(k_x)=i\alpha\omega\langle\lambda^{-2}(\lambda J_l'S_l+J_lR_l)\rangle, \tag{22}$$

which is obtained by multiplying Eq. (20) with $e_m/(\epsilon_Mk_x^2)$ and then using the definitions $\lambda^2\equiv 2p_gk_x^2/m\Omega_m$ and $\alpha\equiv\omega_m^2/(|\epsilon_M(\omega)|\Omega_m^2)$ (with $\omega=l\Omega_m$).

To obtain an explicit analytical solution for Eqs. (19)–(21), we again use the model distribution (11): $F_0(p_g)\sim\delta(p_g-p_0)$, so that Eq. (22) becomes

$$e_m\phi(k_x)=\frac{i\alpha\omega}{\lambda^2}\,[\lambda J_l'S_l(k_x)+J_lR_l(k_x)], \tag{23}$$

where $\lambda=k_x\rho_0$, with $\rho_0=(2p_0/m\Omega_m)^{1/2}$, while Eqs. (19) and (21) are now evaluated at $p_g=p_0$:

$$\left(\frac{\omega}{L}\right)\frac{\partial S_l}{\partial k_x}\,(k_x)=e_mJ_l(\lambda)\,\phi(k_x), \tag{24}$$

$$\left(\frac{\omega}{L}\right)\frac{\partial R_l}{\partial k_x}\,(k_x)=\lambda J_l'(\lambda)e_m\phi(k_x), \tag{25}$$

where, in Eqs. (24)–(25), $S_l(k_x)\equiv S_l(p_0;k_x)$ and $R_l(k_x)\equiv R_l(p_0;k_x)$. Within this model, gyroresonance crossings occur either at zeros of $J_l(\lambda)$ (where the wave-energy sign goes from negative to positive) or at zeros of $J_l'(\lambda)$ (where the wave-energy sign goes from positive to negative). In this section, we consider gyroresonance crossings of the first type; the gyroresonance crossings of the second type are analyzed in a similar fashion, and the results are summarized in Appendix B.

In the neighborhood of a zero of J_l, i.e., $J_l(\lambda_0)=0$ with $\lambda_0=k_0\rho_0$, we Taylor expand the Bessel function as $J_l(\lambda)=(\lambda-\lambda_0)J_l'(\lambda_0)$, and $J_l'(\lambda)=J_l'(\lambda_0)$. When substituted into the coupled equations (24)–(25), using Eq. (23), they become

$$\left(i\,\frac{d}{dq}+q\right)S+q^2R=0,$$
$$\left(i\,\frac{d}{dq}+q\right)R+S=0, \tag{26}$$

where

$$q\equiv\xi^{1/2}(k_x-k_0)/k_0,\quad\text{with}\quad\xi\equiv\alpha k_0L|J_l'(\lambda_0)|^2, \tag{27}$$

is a *dimensionless* variable that vanishes at the gyroresonance crossing, while $S(q)\equiv S_l(k_x)$, $R(q)\equiv \xi^{-1/2}R_l(k_x)$. Notice that, within our model (where $k_z=0=k_y$), no parameters appear in the coupled equations (26). Moreover, from Eq. (23), the self-consistent electrostatic potential is

$$\phi(k_x)\sim S(q)+qR(q), \tag{28}$$

and $\beta(k_x)\sim q$, so that, near the gyroresonance crossing (at $q=0$), the Bernstein-wave energy $W\sim|S+qR|^2/q$ is negative for $q<0$ (positive for $q>0$).

We write the coupled equations (26) in matrix form:

$$\hat{\mathsf{D}}\cdot\psi(q)=0, \tag{29}$$

where the operator matrix

$$\hat{\mathsf{D}}\equiv \mathsf{D}(\hat{p},q)\equiv\begin{pmatrix}\hat{p}+q & q^2\\ 1 & \hat{p}+q\end{pmatrix}, \tag{30}$$

with $\hat{p}\equiv i d/dq$, operates on the two-component field,

$$\boldsymbol{\psi}(q)\equiv\begin{pmatrix}S(q)\\ R(q)\end{pmatrix}. \tag{31}$$

To analyze Eq. (29), we first consider the eikonal representation:

$$\begin{pmatrix}S(q)\\ R(q)\end{pmatrix}=\begin{pmatrix}\tilde{S}(q)\\ \tilde{R}(q)\end{pmatrix}e^{i\Theta(q)}, \tag{32}$$

for a normal mode, where $\tilde{S}$ and $\tilde{R}$ are slowly varying functions of q. With id/dq acting only on the phase factor $\exp i\Theta$, we have $\hat{p}=id/dq\rightarrow -d\Theta/dq\equiv p(q)$ in (30). Then the operator $\hat{\mathsf{D}}=\mathsf{D}(\hat{p},q)$ is replaced by its *symbol* $\mathsf{D}(p,q)$, where (p,q) are now phase-space coordinates. The solution of Eq. (29) thus requires

$$0=\det\,\mathsf{D}(p,q)=p(p+2q), \tag{33}$$

yielding the two *dispersion relations*,

$$p_G(q)=0 \quad\text{and}\quad p_B(q)=-2q. \tag{34}$$

Noting that q is essentially k_x, we see that p is essentially x. In fact, $x(k_x)=-d\Theta/dk_x=-(d\Theta/dq)(dq/dk_x)\equiv p\;dq/dk_x$, i.e., $dx\;dk_x=dp\;dq$, representing a canonical transformation to dimensionless variables. Furthermore, the ray velocity $\dot{q}$ is positive, since $\dot{q}\sim\dot{k}_x=\omega/L>0$.

We now can interpret the dispersion relation (or ray orbit) $p_B(q)=-2q$ of Eq. (34) as that of the Bernstein wave, namely, the local form of Eq. (10). The other mode represents a perturbed distribution with vanishing charge density and self-consistent potential, and is called the *gyroballistic mode*.[2] Since it is carried only by resonant particles, its orbit is $x(k_x)=0$, i.e., $p_G(q)=0$. The respective ray orbits are shown in Fig. 2. At $q=0$, the rays cross, and linear conversion occurs. Near the crossing, the eikonal approximation is invalid, and the coupled equations (26) must be solved exactly.

But first, we express the wave field (S,R) in terms of the two (eikonal) normal modes. From Eqs. (29)–(32), we have $\tilde{S}(q)=-[p(q)+q]\tilde{R}(q)$, so for the Bernstein wave (B), we have [from Eq. (34)] $\tilde{S}_B/\tilde{R}_B=q$, while for the gyroballistic mode (G), we have $\tilde{S}_G/\tilde{R}_G=-q$ [i.e., from Eq. (28), $\tilde{\phi}_G\equiv 0$]. This motivates the transformation $(S,R)\rightarrow(B,G)$:

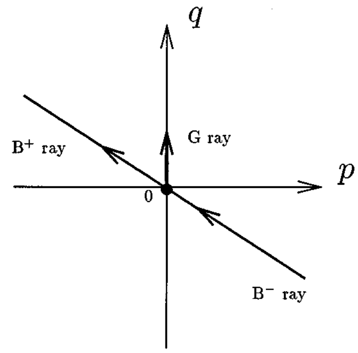

FIG. 2. Linear conversion of a negative-energy Bernstein (B^-) wave into a positive-energy Bernstein (B^+) wave and a negative-energy gyroballistic (G) wave.

$$\begin{pmatrix}S(q)\\ R(q)\end{pmatrix}=\begin{pmatrix}q & -q\\ 1 & 1\end{pmatrix}\begin{pmatrix}B(q)\\ G(q)\end{pmatrix}, \tag{35}$$

where [inverting Eq. (35)]

$$B(q)\equiv(S+qR)/2q \quad\text{and}\quad G(q)\equiv-(S-qR)/2q \tag{36}$$

represent the two normal modes. By substituting (35) into (26), the mode-coupling equations for $B(q)$ and $G(q)$ are (after some minor manipulations)

$$\begin{aligned}\left(2iq\,\frac{d}{dq}+(i+4q^2)\right)B-iG=0,\\ \left(2iq\,\frac{d}{dq}+i\right)G-iB=0.\end{aligned} \tag{37}$$

The advantage gained in working with these coupled equations, instead of the coupled equations (26), is that the coupled equations (37) have a variational formulation in terms of a Hermitian operator (as shown below).

In this section, we have derived the coupled equations (37) near a crossing of the gyroresonance layer where the Bernstein-wave energy changes its sign from negative to positive. In the next section, we show that these coupled equations can be expressed in terms of a variational principle and that the transfer of energy between the two modes obeys an exact energy-conservation law.

IV. VARIATIONAL PRINCIPLE AND CONSERVATION LAW

It is useful to formulate the coupled equations (37) as a variational principle $\delta\mathscr{A}=0$, where the action functional is

$$\mathscr{A}(B,G)\equiv\int dq\;\boldsymbol{\Psi}^\dagger\cdot\hat{\bar{\mathsf{D}}}\cdot\boldsymbol{\Psi}, \tag{38}$$

with a manifestly Hermitian operator,

$$\hat{\bar{\mathsf{D}}}\equiv\bar{\mathsf{D}}(\hat{p},q)\equiv\begin{pmatrix}\hat{p}q+q\hat{p}+4q^2 & -[\hat{p},q]=-i\\ [\hat{p},q]=i & -(\hat{p}q+q\hat{p})\end{pmatrix}, \tag{39}$$

operating on the two-component field,

$$\boldsymbol{\Psi}(q)\equiv\begin{pmatrix}B(q)\\ G(q)\end{pmatrix}. \tag{40}$$

Note that, in the eikonal limit, where $\hat{p}\to p(q)$, the off-diagonal elements $\pm[\hat{p},q]$ vanish, and the matrix (39) is diagonal.

From the action functional (38): $\mathscr{A}\equiv\int dq\ \mathscr{L}$, we obtain the Lagrangian density,

$$\mathscr{L}\left(B,\ \frac{dB}{dq};G,\ \frac{dG}{dq};q\right)$$
$$=iq\left(B^*\ \frac{dB}{dq}-B\ \frac{dB^*}{dq}\right)-iq\left(G^*\ \frac{dG}{dq}-G\ \frac{dG^*}{dq}\right)$$
$$+4q^2|B|^2+i(BG^*-B^*G). \tag{41}$$

If we use the convenient notation $\Psi_1=B$ and $\Psi_2=G$, the Lagrangian density (41) can be used to show that the mode-coupling equations (37) can be written in the Euler–Lagrange form (for $a=1$ or 2):

$$\frac{d}{dq}\left(\frac{\partial\mathscr{L}}{\partial(d\Psi_a^*/dq)}\right)=\frac{\partial\mathscr{L}}{\partial\Psi_a^*}. \tag{42}$$

From the invariance of the Lagrangian density (41) under an infinitesimal constant phase shift:[9] $\Psi_a\to e^{i\varphi}\Psi_a$ (where the constant phase φ is real), we obtain the identity

$$0=\delta\mathscr{L}\equiv\delta\Psi_a\ \frac{\partial\mathscr{L}}{\partial\Psi_a}+\delta\left(\frac{d\Psi_a}{dq}\right)\ \frac{\partial\mathscr{L}}{\partial(d\Psi_a/dq)}+\text{c.c.},$$

where summation over a is implied. Substituting $\delta\Psi_a=i\varphi\Psi_a$ (for $a=1$ and 2) into the expression for $\delta\mathscr{L}$, we find

$$\delta\mathscr{L}=-2\varphi\ \mathrm{Im}\left(\Psi_a\ \frac{\partial\mathscr{L}}{\partial\Psi_a}+\frac{d\Psi_a}{dq}\ \frac{\partial\mathscr{L}}{\partial(d\Psi_a/dq)}\right). \tag{43}$$

When we substitute the Euler–Lagrange equation (42) into Eq. (43) and require that $\delta\mathscr{L}$ vanish for arbitrary φ, we then obtain the conservation law for wave energy:

$$\frac{d\Gamma(q)}{dq}=0, \tag{44}$$

where

$$\Gamma(q)\equiv\mathrm{Im}\left(\Psi_a\ \frac{\partial\mathscr{L}}{\partial(d\Psi_a/dq)}\right)=q|B(q)|^2-q|G(q)|^2 \tag{45}$$

is (proportional to) the flux of energy density in the k_x direction. By direct substitution of the coupled equations (37) into Eq. (44), one easily checks that Eq. (45) is indeed an invariant. Since $\phi(q)\sim qB(q)$ [from (28) and (36)], the first term is $\sim|\phi|^2/q$, and thus corresponds to the Bernstein-wave energy density W near the gyroresonance crossing. For an interpretation of the gyroballistic term, we note that $\Gamma(q)$ can, by (36), be written $\Gamma=\mathrm{Re}(R^*S)$ consistently with (16): for $\phi=0$ (i.e., $S=-qR$ or $B=0$), it becomes $\Gamma(q)=-|S|^2/q$. Hence, for $q<0$, the Bernstein wave has negative energy while the gyroballistic mode has positive energy. As the Bernstein wave crosses the gyroresonance layer (at $q=0$), its energy becomes positive. To conserve energy, the Bernstein wave transfers energy to the gyroballistic mode (which then acquires negative energy).

In this section, the coupled equations (37) for the Bernstein field $B(q)$ and the gyroballistic field $G(q)$ were derived from a variational principle [based on the action integral (38)], and the exact energy conservation law (44) was found. In the next section, we solve the coupled equations (37) and use the results to obtain a relation between the asymptotic forms of the wave fields on each side of the gyroresonance layer. This relation will then yield the conversion and transmission coefficients for the mode conversion process involving the energy transfer from the Bernstein mode to the gyroballistic mode.

V. MODE CONVERSION PROCESS

Before solving the coupled equations (37) exactly, we examine their *asymptotics* and formulate the S matrix for transmission and conversion. For the Bernstein mode, we write

$$B(q)=\tilde{B}(q)e^{i\Theta_B(q)}\quad(|q|\gg 1), \tag{46}$$

where $\Theta_B(q)=-\int p_B(q)dq=q^2$ [by Eq. (34)]. From Eq. (45), with $G\to 0$ for a pure Bernstein mode, we have

$$\tilde{B}(q)\to\begin{cases}B_+/\sqrt{q}, & q\gg 1,\\ B_-/\sqrt{-q}, & q\ll -1,\end{cases} \tag{47}$$

where B_+ and B_- are the *constant* (complex) action amplitudes. Similarly, for the gyroballistic mode, we have

$$G(q)=\tilde{G}(q)e^{i\Theta_G(q)}\quad(|q|\gg 1), \tag{48}$$

where $\Theta_G(q)=-\int p_G(q)dq=0$ and

$$\tilde{G}(q)\to\begin{cases}G_+/\sqrt{q}, & q\gg 1,\\ G_-/\sqrt{-q}, & q\ll -1.\end{cases} \tag{49}$$

We wish to determine the relation between the outgoing action amplitudes (B_+,G_+) and the incident amplitudes (B_-,G_-):

$$\begin{pmatrix}B_+\\ G_+\end{pmatrix}=\begin{pmatrix}S_{BB} & S_{BG}\\ S_{GB} & S_{GG}\end{pmatrix}\begin{pmatrix}B_-\\ G_-\end{pmatrix}. \tag{50}$$

Introducing the action amplitude vectors for the outgoing $(q>0)$ and incident $(q<0)$ fields,

$$\mathbf{A}_\pm\equiv\begin{pmatrix}B_\pm\\ G_\pm\end{pmatrix}, \tag{51}$$

we write Eq. (50) as

$$\mathbf{A}_+=\mathsf{S}\cdot\mathbf{A}_- . \tag{52}$$

In the eikonal regions, the (constant) energy flux (45) takes the form

$$\Gamma=\mathbf{A}^\dagger\cdot\boldsymbol{\sigma}\cdot\mathbf{A}\ \mathrm{sgn}\ q,\quad \boldsymbol{\sigma}\equiv\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{53}$$

with $\Gamma_+=\Gamma_-$. Inserting Eq. (52) into Eq. (53) for Γ_+, and noting that $\mathbf{A}_-$ is arbitrary, we obtain the condition

$$\mathsf{S}^\dagger\cdot\boldsymbol{\sigma}\cdot\mathsf{S}=-\boldsymbol{\sigma}, \tag{54}$$

on the S matrix. (This replaces the conventional unitary condition $\mathsf{S}^\dagger\cdot\mathsf{S}=\mathbf{I}$.)

Because the coupled equations (37) are invariant under the parity operation, they have two solutions of definite parity, even in q and odd in q, respectively. For the even solution $\mathbf{A}^e$, we have $\mathbf{A}_+^e = \mathbf{A}_-^e$, so $\mathbf{A}_-^e$ is an eigenvector of S with eigenvalue 1. Likewise, for the odd solution $\mathbf{A}^o$, we have $\mathbf{A}_+^o = -\mathbf{A}_-^o$, so $\mathbf{A}_-^o$ is an eigenvector of S with eigenvalue -1. From its eigenvalues, we see that $\mathrm{Tr}\,\mathsf{S}=0$ and $\det\mathsf{S}=-1$. These two conditions based on parity, with Eq. (54) based on energy conservation, yields (by elementary algebra) the following form for S:

$$\mathsf{S}=\begin{pmatrix} ia & (1+a^2)^{1/2}e^{i\gamma} \\ (1+a^2)^{1/2}e^{-i\gamma} & -ia \end{pmatrix}, \tag{55}$$

where a and γ (both real) are still to be determined from the polarizations G_-/B_- of the two eigenvectors.

The even and odd solutions of Eqs. (37), for all q, are expressible in terms of the confluent hypergeometric function:[10]

$$\begin{aligned} B^e(q) &= M(\tfrac{5}{4},\tfrac{3}{2},iq^2), \\ G^e(q) &= -[B^e(q)e^{-iq^2}]^*, \\ B^o(q) &= q^{-1}M(\tfrac{3}{4},\tfrac{1}{2},iq^2), \\ G^o(q) &= [B^o(q)e^{-iq^2}]^*. \end{aligned} \tag{56}$$

From the known asymptotics of this function,[10] we obtain the required polarizations:

$$\begin{aligned} G_-^e/B_-^e &= -e^{i\pi/4}, \\ G_-^o/B_-^o &= e^{-i\pi/4}. \end{aligned} \tag{57}$$

Again elementary algebra, using (57), yields $a=-1$ and $\gamma=\pi$, so the S matrix is

$$\mathsf{S}=\begin{pmatrix} -i & -\sqrt{2} \\ -\sqrt{2} & +i \end{pmatrix}. \tag{58}$$

Thus, the amplitude transmission coefficient for the Bernstein wave is $S_{BB}=-i$, and for the gyroballistic mode is $S_{GG}=+i$. The amplitude conversion coefficients are $S_{BG}=S_{GB}=-\sqrt{2}$. The energy transmission coefficients are

$$\begin{aligned} T_B &\equiv -|S_{BB}|^2 = -1, \\ T_G &\equiv -|S_{GG}|^2 = -1, \end{aligned} \tag{59}$$

where the minus sign (used in the definitions) indicates that the energy sign has flipped, while the energy conversion coefficients are

$$\begin{aligned} C_{BG} &\equiv |S_{BG}|^2 = +2, \\ C_{GB} &\equiv |S_{GB}|^2 = +2. \end{aligned} \tag{60}$$

Thus, for an incident (negative-energy) Bernstein wave of energy $W_B^-=-1$, the transmitted Bernstein wave has energy $W_B^+=T_B W_B^-=+1$, while the conversion to the gyroballistic mode has energy $W_G^+=C_{GB}W_B^-=-2$. (Energy conservation thus reads as $-1=+1-2$.)

VI. SUMMARY

We have shown that a Bernstein (collective) wave supported by an inverted minority-ion population can, in a nonuniform magnetic field, change the sign of its energy when it crosses the gyroresonance layer. It does so by transferring energy through linear mode conversion with a gyroballistic (noncollective) mode, which exists in the gyroresonance layer.

Using a simple model distribution for the energetic minority ions, we obtained a set of two coupled equations near the gyroresonance layer, which we solved exactly. An exact conservation law was also derived and shown to correspond to energy conservation. From the asymptotic properties of these solutions, we constructed an S-matrix relation between the eikonal amplitudes on both sides of the resonance layer. This relation was then used to obtain the explicit conversion and transmission coefficients: $C=2$ and $T=-1$.

ACKNOWLEDGMENTS

We thank R. G. Littlejohn for his technical advice and W. B. Kunkel for his insightful comments.

This work was supported by the U.S. Department of Energy under Contract No. DE-AC03-76SF00098.

APPENDIX A: LINEARIZED VLASOV–POISSON EQUATIONS FOR THE MINORITY-ION BERNSTEIN WAVE

In this appendix, we derive the coupled equations for the minority perturbation amplitude $\tilde{S}_l$ and the potential amplitude $\tilde{\phi}$ in a nonuniform magnetic field $\mathbf{B}(x)=\hat{z}(1-x/L)B_0$.

The Vlasov equation for the minority distribution f is

$$0=\frac{df}{dt}=\frac{\partial f}{\partial t}+\{f,h\}, \tag{A1}$$

where h is the particle Hamiltonian and { , } is the Poisson bracket on pairs of phase-space functions. Linearizing Eq. (A1), with $f=f_0+\delta f$ and $h=h_0+\delta h$, we have

$$\frac{d^{(0)}\delta f}{dt}\equiv\frac{\partial\delta f}{\partial t}+\{\delta f,h_0\}=-\{f_0,\delta h\}. \tag{A2}$$

In the guiding-center representation,[5] with phase-space coordinates $(\theta,p_g;Z,p_z;X,Y)$, we have

$$\frac{d^{(0)}\delta f}{dt}=\frac{\partial\delta f}{\partial t}+\dot{\theta}^{(0)}\,\frac{\partial\delta f}{\partial\theta}, \tag{A3}$$

where $\dot{p}_g^{(0)}=0$, $\dot{p}_z^{(0)}=0$, and $\dot{X}^{(0)}=0$, while $\partial\delta f/\partial Z=0$ and $\partial\delta f/\partial Y=0$ (since $k_z=0=k_y$). Focusing on the lth-harmonic minority-ion gyroresonance ($\omega\approx l\Omega_m$), we write $\delta f(\theta,p_g;p_z;X,t)=\delta f_l(p_g,p_z;X)\exp(il\theta-i\omega t)$, and Eq. (A3) becomes

$$\frac{d^{(0)}\delta f}{dt}=-i[\omega-l\Omega_m(X)]\delta f. \tag{A4}$$

It is convenient to express δf in terms of the perturbation *generator* S:[6]

$$\delta f=\{S,f_0\}. \tag{A5}$$

From Eqs. (A2), (A4), and (A5), using $\{\Omega_m(X), f_0(p_g,p_z,X)\}=0$, we then have

$$-i[\omega-l\Omega_m(X)]S=\delta h. \tag{A6}$$

In Eq. (A6), we have (in the guiding-center representation)

$$\delta h(\mathbf{r}=\mathbf{R}+\boldsymbol{\rho};p_z,p_g,\theta)$$

$$=e_m\int d^3x\;\delta^3[\mathbf{x}-(\mathbf{R}+\boldsymbol{\rho})]\phi(\mathbf{x}), \tag{A7a}$$

$$=e_m\int d^3x\;\delta^3[\mathbf{x}-(\mathbf{R}+\boldsymbol{\rho})]\tilde{\phi}(\mathbf{x})e^{i\Theta(\mathbf{x})}, \tag{A7b}$$

$$=e_m\tilde{\phi}(\mathbf{R}+\boldsymbol{\rho})e^{i\Theta(\mathbf{R}+\boldsymbol{\rho})}, \tag{A7c}$$

$$\approx e_m\tilde{\phi}(\mathbf{R})\exp[i\Theta(\mathbf{R})+i\mathbf{k}(\mathbf{R})\cdot\boldsymbol{\rho}(p_g,\theta;\mathbf{R})], \tag{A7d}$$

where $\mathbf{R}=(X,Y,Z)$ is the guiding-center position and $\boldsymbol{\rho}(p_g,\theta;\mathbf{R})$ is the gyroradius vector, while the eikonal amplitude $\tilde{\phi}(\mathbf{x})$ and the wave vector $\mathbf{k}(\mathbf{x})=\boldsymbol{\nabla}\Theta(\mathbf{x})$ are slowly varying functions of $\mathbf{x}$. Using $\mathbf{k}=\hat{x}k_x$ and $\boldsymbol{\rho}=(\hat{x}\sin\theta-\hat{y}\cos\theta)\rho$, we have $\mathbf{k}\cdot\boldsymbol{\rho}=\lambda\sin\theta$, where $\lambda=k_x(X)\rho(p_g,X)$, and substituting the identity $e^{i\lambda\sin\theta}=\Sigma_l J_l(\lambda)e^{il\theta}$ into Eq. (A7d), we find $\delta h(\theta,p_g;X)=\Sigma_l\;\delta\tilde{h}_l(p_g,X)\exp[il\theta+i\Theta(X)]$, where $\delta\tilde{h}_l(p_g,X)=e_m\tilde{\phi}(X)J_l(\lambda)$, and Eq. (A6) is replaced with

$$-i[\omega-l\Omega_m(X)]\tilde{S}_l(p_g,X)=e_m\tilde{\phi}(X)J_l[\lambda(p_g,X)]. \tag{A7e}$$

This evolution equation for $\tilde{S}_l$ is equivalent to the linearized Vlasov equation (A2).

With the majority treated as a dielectric, the Poisson equation for $\phi(\mathbf{x})e^{-i\omega t}$ is

$$-\frac{e_M(\omega)}{4\pi}\nabla^2\phi(\mathbf{x})=\rho_m(\mathbf{x}), \tag{A8}$$

where the minority (perturbed) charge density is

$$\rho_m(\mathbf{x})=e_m\int d^6z\;\delta^3[\mathbf{x}-\mathbf{r}(\mathbf{z})]\delta f(\mathbf{z}). \tag{A9}$$

In the guiding-center representation,[5] $d^6z=[e_mB(\mathbf{R})/c]\times d^3R\,d\theta\,dp_g\,dp_z$, the particle position is $\mathbf{r}(\mathbf{z})=\mathbf{R}+\boldsymbol{\rho}(\theta,p_g,\mathbf{R})$, and [from Eq. (A5)]

$$\delta f(\mathbf{z})=il\tilde{S}_l(\mathbf{R},p_g)e^{il\theta+i\Theta(\mathbf{R})}\;\frac{\partial f_0}{\partial p_g}(\mathbf{R},p_g,p_z). \tag{A10}$$

Substituting these expressions into Eq. (A9) and performing the d^3R integration, we have

$$\rho_m(\mathbf{x})=\left(\frac{e_m^2}{c}\right)\int d\theta\,dp_g\,dp_z\,B(\mathbf{x}-\boldsymbol{\rho})\left|\det\left(\mathbf{I}+\frac{\partial\boldsymbol{\rho}}{\partial\mathbf{R}}\right)\right|^{-1}$$
$$\times\delta f(\mathbf{x}-\boldsymbol{\rho};\theta,p_g,p_z), \tag{A11}$$

where [from Eq. (A10)]

$$\delta f(\mathbf{x}-\boldsymbol{\rho};p_g,\theta,p_z)=il\tilde{S}_l(\mathbf{x}-\boldsymbol{\rho},p_g)e^{il\theta}e^{i\Theta(\mathbf{x}-\boldsymbol{\rho})}$$
$$\times\frac{\partial f_0}{\partial p_g}(\mathbf{x}-\boldsymbol{\rho},p_g,p_z).$$

In the eikonal limit, we replace in Eq. (A11): $\det\to1$, $\tilde{S}_l(\mathbf{x}-\boldsymbol{\rho},p_g)\to\tilde{S}_l(\mathbf{x},p_g)$, $f_0(\mathbf{x}-\boldsymbol{\rho},p_g,p_z)\to f_0(\mathbf{x},p_g,p_z)$, $B(\mathbf{x}-\boldsymbol{\rho})\to B(\mathbf{x})$, and $\Theta(\mathbf{x}-\boldsymbol{\rho})\to\Theta(\mathbf{x})-\mathbf{k}(\mathbf{x})\cdot\boldsymbol{\rho}(\theta,p_g,\mathbf{x})$. Thus

$$\rho_m(\mathbf{x})=ile_me^{i\Theta(\mathbf{x})}\int dp_g\,\tilde{S}_l(\mathbf{x},p_g)\;\frac{\partial F_0}{\partial p_g}(\mathbf{x},p_g)$$
$$\times\left(\int\frac{d\theta}{2\pi}\exp(il\theta-i\lambda\sin\theta)\right), \tag{A12}$$

where $F_0(\mathbf{x},p_g)\equiv[2\pi e_mB(\mathbf{x})/c]\int dp_zf_0(\mathbf{x},p_g,p_z)$. The θ-integral equals $J_l(\lambda)$ and Eq. (A12) becomes

$$\rho_m(\mathbf{x})=ile_me^{i\Theta(\mathbf{x})}\int dp_g\;\tilde{S}_l(\mathbf{x},p_g)J_l[\lambda(\mathbf{x},p_g)]$$
$$\times\frac{\partial F_0}{\partial p_g}(\mathbf{x},p_g). \tag{A13}$$

Using the eikonal representation for $\phi(\mathbf{x})$, we thus obtain from Eq. (A8),

$$\frac{\epsilon_M(\omega)}{4\pi}k_x^2(x)\tilde{\phi}(x)$$
$$=ile_m\int dp_g\;\tilde{S}_l(x,p_g)J_l[\lambda(x,p_g)]\frac{\partial F_0}{\partial p_g}(x,p_g). \tag{A14}$$

APPENDIX B: GYRORESONANCE CROSSINGS OF THE SECOND TYPE

Gyroresonance crossings of the second type are defined in Sec. III as crossings for which the sign of the Bernstein-wave energy goes from positive to negative. Within our present model, using Eqs. (10)–(12), these crossings correspond to zeros of J_l'.

In the neighborhood of a zero of J_l', i.e., $J_l'(\lambda_1)=0$ with $\lambda_1=k_1\rho_0$, we Taylor expand the Bessel function as $J_l(\lambda)=J_l(\lambda_1)$ and $J_l'(\lambda)=(\lambda-\lambda_1)J_l''(\lambda_1)\equiv-(\lambda-\lambda_1)\times|J_l'(\lambda_1)/J_l(\lambda_1)|J_l(\lambda_1)$. (In the last expression, we have used the fact that, when $J_l'=0$, $J_l''/J_l<0$.) When substituted into Eq. (26), using Eq. (23), we obtain the following coupled equations:

$$\begin{aligned}\left(i\frac{d}{dr}-r\right)S+R&=0,\\ \left(i\frac{d}{dr}-r\right)R+r^2S&=0,\end{aligned} \tag{B1}$$

where $r\equiv\zeta^{1/2}(k_x-k_1)/k_1$ is a dimensionless variable that vanishes at the gyroresonance crossing, with $\zeta\equiv\alpha k_1L|J_l''J_l|$, while $S(r)\equiv\zeta^{-1/2}(\lambda_1^2|J_l''/J_l|)S_l(k_x)$ and $R(r)\equiv R_l(k_x)$. In addition, the self-consistent electrostatic potential is $\phi\sim(R-rS)$.

In the eikonal limit ($|r|\gg1$), the operator $\hat{s}=id/dr$ becomes $s(r)$, and a treatment similar to Eqs. (31)–(34) yields the two dispersion relations: $s_G(r)=0$, corresponding to the gyroballistic mode, and $s_B(r)=2r$, corresponding to the Bernstein mode.

A comparison of Eqs. (26) and (B1) shows that the transformation $(S,R)\to(-R^*,S^*)$ relates the solutions of Eq. (B1) with the solutions of Eq. (26). As a result, the functions B and G transform as $B\to-B^*=-(R^*-rS^*)/2r$ and $G\to-G^*=(R^*+rS^*)/2r$. Hence, it can easily be checked that the S matrix transforms as $\mathsf{S}\to\mathsf{S}^*$, so that the conversion and transmission coefficients for gyroresonance crossings of the second type are $C=2$ and $T=-1$.

[1] G. A. Cottrell and R. O. Dendy, Phys. Rev. Lett. **60**, 33 (1988); G. A. Cottrell, V. P. Bhatnagar, O. Da Costa, R. O. Dendy, J. Jacquinot, K. G. McClements, D. C. McCune, M. F. F. Nave, P. Smeulders, and D. F. H. Start, Nucl. Fusion **33**, 1365 (1993).
[2] H. Ye and A. N. Kaufman, Phys. Fluids B **4**, 1735 (1992).
[3] L. V. Korablev, Sov. Phys. JETP **26**, 922 (1968); T. D. Kaladze and A. B. Mikhailovskii, Sov. J. Plasma Phys. **1**, 128 (1975).
[4] R. O. Dendy, C. N. Lashmore-Davies, K. G. McClements, and G. A. Cottrell, Phys. Plasmas **1**, 1918 (1994); R. O. Dendy, Plasma Phys. Controlled Fusion **36**, B163 (1994).
[5] R. G. Littlejohn, J. Plasma Phys. **29**, 111 (1983).
[6] P. J. Morrison and D. Pfirsch, Phys. Fluids B **2**, 1105 (1990), Appendix B.
[7] From Ref. 3, a realistic neonatal distribution has the form $f_0(p_\parallel, p_g) \sim \delta(\mathscr{E} - \mathscr{E}_0)$. Thus, the gyromomentum distribution $F_0(p_g) \equiv \int dp_\parallel \, f_0(p_\parallel, p_g) \sim (p_0 - p_g)^{-1/2}$ is inverted [i.e., $F_0'(p_g) > 0$ for $p_g > 0$], and $\beta(k_x) \equiv \langle \lambda^{-1} \, dJ_l^2/d\lambda \rangle$ becomes $\beta(k_x) = J_{2l}(2\lambda_0)/\lambda_0^2$, where $\lambda_0 = k_x \rho(p_0)$. Hence, $\partial\epsilon/\partial\omega \sim \beta^{-1}$ again changes sign by going through infinity as β goes through zero. The gyroresonance crossings are now identified with the zeros of $J_{2l}(2\lambda_0)$.
[8] J. R. Cary, Phys. Rep. **79**, 129 (1981).
[9] A. J. Brizard and A. N. Kaufman, Phys. Rev. Lett. **74**, 4167 (1995).
[10] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1965), Chap. 13.