How a Wave Flips Its Energy Sign by Linear Conversion

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An inverted population of energetic minority ions, such as neonatal fusion alphas, can support a negative-energy Bernstein wave (whose frequency is a harmonic of their gyrofrequency). As a result of the magnetic-field nonuniformity and the population inversion, the wave crosses the gyroresonance layer and its energy flips sign. This results in energy transfer to a gyroballistic mode (which exists only in the resonance layer), with a conversion coefficient exactly equal to 2.

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Negative-energy Bernstein waves, supported by an *inverted* population of energetic minority ions (such as charged fusion products), have recently attracted attention [1] because of their suspected role in the enhanced emission at harmonics of the minority-ion gyrofrequency observed in several laboratory and space magnetized plasmas. In this Letter, we report on our study of their propagation in a *nonuniform* magnetic field; a complete description of this work is presented in Ref. [2].

We show that wave refraction, i.e., $\dot{\mathbf{k}} \equiv -\nabla\omega(\mathbf{k}, \mathbf{x})$, causes the energy of a Bernstein wave supported by an inverted population (which is $\mathbf{k}$ dependent) to change *sign* when the wave crosses the gyroresonance layer. This situation presents us with the following paradox: How can a wave change the sign of its energy while maintaining overall energy conservation? We resolve this paradox by showing that the Bernstein wave changes its energy sign by transferring energy to a gyroballistic mode [3] which exists only in the gyroresonance layer. (A ballistic mode represents a perturbed distribution with vanishing charge density and self-consistent potential.)

In order to get explicit analytical results, we make simplifying assumptions concerning the unperturbed energetic minority-ion distribution. For the neonatal charged fusion products, we choose the model distribution [4]

$$f_0(p_g) \sim \delta(p_g - p_0), \tag{1}$$

which significantly simplifies our analysis; here p_g is the gyromomentum of the minority ions (the gyroangle θ_g is canonically conjugate), and $p_0 = \mathcal{E}_0/\Omega$ is the gyromomentum corresponding to the birth energy $\mathcal{E}_0$. We use a slab model for the background magnetic field $\mathbf{B}(x) \equiv \hat{z}(1 - x/L)B_0$, where $x = 0$ is the location of the lth-harmonic minority gyroresonance layer: $\omega = l\Omega_{\rm m}(x = 0)$ (with $l \sim 5$). In addition, we choose the wave vector $\mathbf{k} = \hat{x}k_x$, i.e., $k_z = 0$ (often used for Bernstein waves) and $k_y = 0$ (used here for simplicity). We note that, with $k_z = 0$, the parallel-momentum distribution plays no role, and will henceforth be omitted. This choice of wave vector means that minority ions are resonant with the field (at frequency ω) when their *guiding centers* are located at $x = 0$. The corresponding *particles*, however, have a spatial spread given by their gyroradius; hence the latter can be considered the thickness of the gyroresonance, as in the work of Lashmore-Davies and Dendy [5].

We begin with a simple model for a magnetized plasma composed of a "cold" majority-ion (M) fluid (with density $n_{\rm M}$), an electron fluid, and an energetic minority-ion (m) population described by a linearized Vlasov equation (in the guiding-center representation). (The use of the cold-fluid approximation for the majority species is justified for $l \geq 3$, since the thermal effect of the majority ions can be neglected compared to that of the energetic minority ions.) For definiteness, the majority-ion species may be chosen to be deuterium, while energetic minority ions may be alpha particles.

The *local* dielectric function for the energetic minority-ion Bernstein wave is

$$\epsilon(k_x, x; \omega) = \epsilon_{\rm M}(\omega) + \chi_l(k_x, x; \omega), \tag{2}$$

where $\epsilon_{\rm M}(\omega) = -\omega_{\rm M}^2/(\omega^2 - \Omega_{\rm M}^2)$ is the majority dielectric function (in the cold-fluid approximation) and the minority susceptibility is $\chi_l(k_x, x; \omega) = -(\omega_{\rm m}^2/\Omega_{\rm m}^2)\{\omega/[\omega - l\Omega_{\rm m}(x)]\}\beta(k_x)$, with the minority-ion plasma frequency $\omega_{\rm m} = (4\pi n_{\rm m} e_{\rm m}^2/m)^{1/2} \ll \omega_{\rm M}$ and

$$\beta(k_x) \equiv \frac{2}{\lambda} J_l(\lambda) J_l'(\lambda). \tag{3}$$

Here, J_l is a Bessel function with argument $\lambda \equiv k_x\rho_0$, where $\rho_0 = (2p_0/m\Omega_{\rm m})^{1/2}$ is the gyroradius of the minority ions, $J_l' = dJ_l/d\lambda$, and the distribution (1) was used. We note that $\beta(k_x)$ is an *oscillating* function of k_x and, more importantly, it can *change sign* by going through zero (when either J_l or J_l' vanishes).

Solving $\epsilon(k_x, x; \omega) = 0$ for ω, we obtain the *local* dispersion relation

$$\begin{aligned}\omega(k_x, x) &= l\Omega_{\rm m}(x)[1 - \alpha\beta(k_x)] \\ &= \omega\left(1 - \frac{x}{L} - \alpha\beta(k_x)\right),\end{aligned} \tag{4}$$

where $\alpha \equiv (l^2 - \Omega_{\rm M}^2/\Omega_{\rm m}^2)\omega_{\rm m}^2/\omega_{\rm M}^2 \ll 1$ and $|\beta| < 1$. Setting $\omega(k_x, x) = \omega$, we then obtain the ray orbit

$$x(k_x)/L = -\alpha\beta(k_x)\,; \tag{5}$$

the ray velocity (in k_x) is given by the Hamilton equation $\dot{k}_x = -\partial\omega/\partial x = \omega/L > 0$ (see Fig. 1 for this orbit where we take $l = 5$). Note that, from Eq. (5), the ray crosses the gyroresonance layer (at $x = 0$) whenever β vanishes, while the group velocity, given by the Hamilton equation $\dot{x} = \partial\omega/\partial k_x = -\omega\alpha\, d\beta/dk_x$, vanishes when $d\beta(k_x)/dk_x = 0$; from Eq. (3) the ray crosses the layer repeatedly as λ alternately passes through the zeros of J_l and J_l'. (It is important to keep in mind that, as k_x increases, it will eventually reach a value for which the cold-fluid approximation for the majority ions becomes invalid and a more realistic model must be used; under normal conditions, however, several gyroresonance crossings occur before the cold-fluid approximation breaks down.)

The Bernstein-wave energy density is $W(x) \equiv \omega(\partial\epsilon/\partial\omega)k_x^2|\tilde{\phi}|^2/4\pi$, where $\tilde{\phi}$ is the eikonal amplitude of the perturbed electrostatic potential. From Eqs. (2) and (4), we find $\omega\,\partial\epsilon/\partial\omega = |\epsilon_{\rm M}/\alpha|/\beta(k_x)$, and thus the wave energy W has the same sign as β. Referring to Fig. 1, we see that W changes sign (as β goes through zero) whenever the ray crosses the gyroresonance layer at $x = 0$; at the first crossing in Fig. 1 J_l' vanishes and W goes from positive to negative, whereas at the second crossing J_l vanishes and W goes from negative to positive. (The group velocity $\dot{x}$, on the other hand, changes sign as $d\beta/dk_x$ goes through zero.) Note that, as $x \to 0$, $\partial\epsilon/\partial\omega \to \infty$, and so (within the eikonal approximation) the potential amplitude $\tilde{\phi} \to 0$. Of course, in this singular (resonance) region, a noneikonal treatment is required, which we study next.

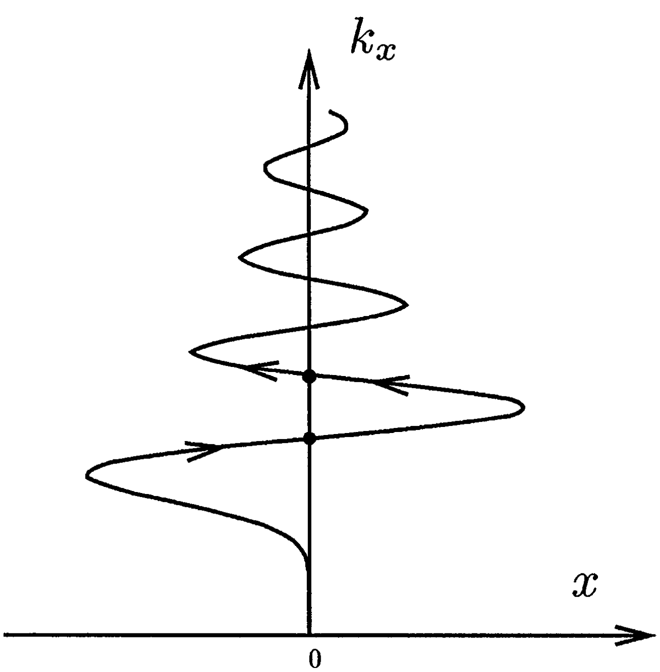

FIG. 1. Plot of the energetic minority-ion Bernstein ray orbit (5) in ray phase space (x, k_x). (See text for comments.)

Near the gyroresonance layer, the Bernstein wave [a collective wave with frequency $\omega \approx l\Omega_{\rm m}(x)$] is strongly coupled to the gyroballistic mode [a ballistic mode which lives inside the resonance layer and satisfies the dispersion relation: $\omega = l\Omega_{\rm m}(x)$]. In the k_x representation (where $x \to \hat{x} = i\partial/\partial k_x$), the perturbed minority-ion distribution (in the guiding-center representation) $\delta f(k_x; p_g, \theta_g; t) = \delta f_l(k_x, p_g)\exp[i(l\theta_g - \omega t)]$ and the perturbed electrostatic potential $\Phi(k_x, t) = \phi(k_x)\exp(-i\omega t)$ satisfy the linearized Vlasov-Poisson equations:

$$[\omega - l\Omega_{\rm m}(\hat{x})]\delta f_l(k_x, p_g) = -le_{\rm m}f_0'(p_g)J_l(k_x\rho_g)\phi(k_x)\,,$$
$$\epsilon_{\rm M}(\omega)k_x^2\phi(k_x) = 4\pi e_{\rm m}\int J_l(k_x\rho_g)\,\delta f_l(k_x, p_g)\,, \tag{6}$$

where $\int \equiv 2\pi\int_0^\infty dp_g$. In the vicinity of a zero of J_l [i.e., for k_x near k_0, where $J_l(k_0\rho_0) = 0$], the fields δf_l and ϕ can be expressed in terms of the *Bernstein-wave* field $B(q)$ and the *gyroballistic-mode* field $G(q)$ as $\delta f_l \sim q(B - G)f_0'$ and $\phi \sim qB$, where $q \sim (k_x - k_0)$. The coupled equations for $B(q)$ and $G(q)$ can be derived from a variational principle [2]:

$$\mathcal{A}(B, G) \equiv \int dq\, \Psi^\dagger \cdot \hat{\mathrm{D}} \cdot \Psi\,, \tag{7}$$

where the Hermitian operator

$$\hat{\mathrm{D}} \equiv \mathrm{D}(\hat{p}, q) \equiv \begin{pmatrix} \hat{p}q + q\hat{p} + 4q^2 & -[\hat{p}, q] \\ [\hat{p}, q] & -(\hat{p}q + q\hat{p}) \end{pmatrix}, \tag{8}$$

with $\hat{p} = id/dq$ and $[\hat{p}, q] = \hat{p}q - q\hat{p} = i$, acts on the two-component field

$$\Psi(q) \equiv \begin{pmatrix} B(q) \\ G(q) \end{pmatrix}. \tag{9}$$

Note that, in the eikonal limit ($|q| \gg 1$), the matrix (8) is *diagonal.* The two eigenvalues (Λ_1, Λ_2) of D are $\Lambda_1(p, q) = 2q[p(q) + 2q]$ and $\Lambda_2(p, q) = -2qp(q)$; the relations $\Lambda_1(p, q) = 0$ and $\Lambda_2(p, q) = 0$ (for $q \neq 0$) yield the dispersion relations

$$p_B(q) = -2q \quad \text{and} \quad p_G(q) = 0 \tag{10}$$

for the Bernstein wave [i.e., it is the local form of Eq. (5)] and for the gyroballistic mode, respectively. The two ray orbits are shown in Fig. 2; at $q = 0$, the rays cross and linear conversion occurs.

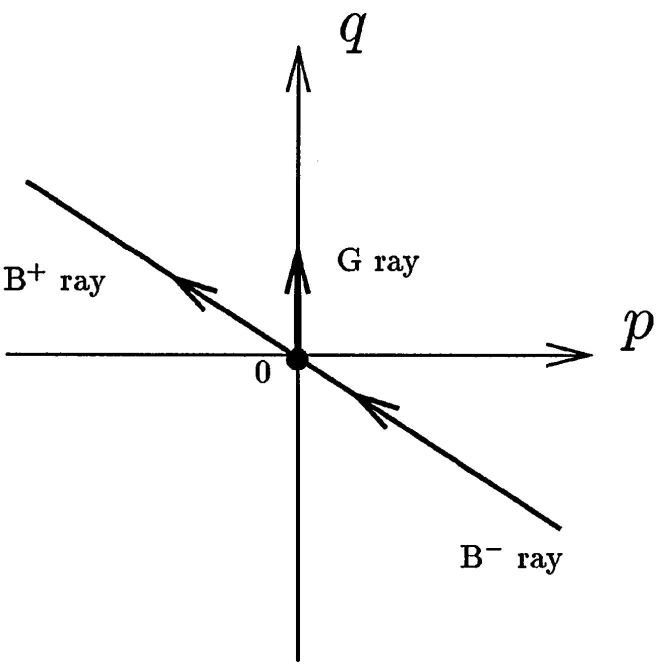

FIG. 2. Linear conversion of a negative-energy Bernstein (B^-) wave into a positive-energy Bernstein (B^+) wave and a negative-energy gyroballistic (G) wave.

On varying (7) with respect to B^* and G^*, we obtain the mode-coupling equations

$$\left(2iq\frac{d}{dq} + (i + 4q^2)\right)B - iG = 0\,,$$
$$\left(2iq\frac{d}{dq} + i\right)G - iB = 0\,. \tag{11}$$

From Noether symmetry of the Lagrangian density [the integrand in (7)] under an infinitesimal constant phase shift [6], we obtain a conservation law for wave energy: $d\Gamma(q)/dq = 0$, where

$$\Gamma(q) \equiv q|B(q)|^2 - q|G(q)|^2 \tag{12}$$

is (proportional to) the flux of energy density in the k_x direction. For $q < 0$ (i.e., before the crossing), the Bernstein wave has negative energy, while the gyroballistic mode has positive energy. As the Bernstein wave crosses the gyroresonance layer (at $q = 0$), its energy becomes positive. To conserve energy, the Bernstein wave transfers energy to the gyroballistic mode (which then acquires negative energy).

Before solving the coupled equations (11) exactly, we examine their *asymptotics* and formulate the S matrix for transmission and conversion. For the Bernstein mode, we write $B(q) = \tilde{B}(q)\exp[i\Theta_B(q)]$ for $|q| \gg 1$, where $\Theta_B(q) = -\int p_B(q)\,dq = q^2$ [by Eq. (10)]. From Eq. (12), with $G \to 0$ for a pure Bernstein mode, we have

$$\tilde{B}(q) \longrightarrow \begin{cases} B_+/\sqrt{q}, & q \gg 1\,, \\ B_-/\sqrt{-q}, & q \ll -1\,, \end{cases} \tag{13}$$

where B_+ and B_- are the *constant* (complex) action amplitudes. Similarly, for the gyroballistic mode, we have $G(q) = \tilde{G}(q)\exp[i\Theta_G(q)]$ for $|q| \gg 1$, where $\Theta_G(q) = -\int p_G(q)\,dq = 0$ and

$$\tilde{G}(q) \longrightarrow \begin{cases} G_+/\sqrt{q}\,, & q \gg 1\,, \\ G_-/\sqrt{-q}\,, & q \ll -1\,. \end{cases} \tag{14}$$

We wish to determine the relation between the outgoing action amplitudes $\mathbf{A}_+ \equiv (B_+, G_+)$ and the incident amplitudes $\mathbf{A}_- \equiv (B_-, G_-)$:

$$\mathbf{A}_+ = \mathrm{S}\cdot\mathbf{A}_-\,. \tag{15}$$

In the eikonal regions, the (constant) energy flux (12) takes the form

$$\Gamma = \mathbf{A}^\dagger\cdot\boldsymbol{\sigma}\cdot\mathbf{A}\,\mathrm{sgn}q, \quad \boldsymbol{\sigma} \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{16}$$

with $\Gamma_+ = \Gamma_-$. Inserting Eq. (15) into Eq. (16) for Γ_+, and noting that $\mathbf{A}_-$ is arbitrary, we obtain the condition

$$\mathrm{S}^\dagger\cdot\boldsymbol{\sigma}\cdot\mathrm{S} = -\boldsymbol{\sigma} \tag{17}$$

on the S matrix. (This replaces the conventional unitary condition $\mathrm{S}^\dagger\cdot\mathrm{S} = \mathbf{I}$.) Because the coupled equations (11) are invariant under the parity operation, they have two solutions of definite parity, even in q and odd in q, respectively. For the even solution $\mathbf{A}^e$, we have $\mathbf{A}^e_+ = \mathbf{A}^e_-$, so $\mathbf{A}^e_-$ is an eigenvector of S with eigenvalue 1. Likewise, for the odd solution $\mathbf{A}^o$, we have $\mathbf{A}^o_+ = -\mathbf{A}^o_-$, so $\mathbf{A}^o_-$ is an eigenvector of S with eigenvalue -1. From its eigenvalues, we see that $\mathrm{TrS} = 0$ and $\det\mathrm{S} = -1$. These two conditions based on parity, with Eq. (17) based on energy conservation, yield (by elementary algebra) the following form for S:

$$\mathrm{S} = \begin{pmatrix} ia & (1 + a^2)^{1/2}e^{i\gamma} \\ (1 + a^2)^{1/2}e^{-i\gamma} & -ia \end{pmatrix}, \tag{18}$$

where a and γ (both real) are still to be determined from the polarizations G_-/B_- of the two eigenvectors.

From the asymptotic properties of the solutions of (11), we obtain the required polarizations [2] and find $a = -1$ and $\gamma = \pi$, so the S matrix is

$$\mathrm{S} = \begin{pmatrix} -i & -\sqrt{2} \\ -\sqrt{2} & +i \end{pmatrix}. \tag{19}$$

Thus the amplitude transmission coefficient for the Bernstein wave is $S_{BB} = -i$, and for the gyroballistic mode

$S_{GG} = +i$. The amplitude conversion coefficients are $S_{BG} = S_{GB} = -\sqrt{2}$. The energy transmission coefficients are

$$T_B \equiv -|S_{BB}|^2 = -1\,, \quad T_G \equiv -|S_{GG}|^2 = -1\,, \tag{20}$$

where the minus sign (used in the definitions) indicates that the energy sign has flipped, while the energy conversion coefficients are

$$C_{BG} \equiv |S_{BG}|^2 = +2\,, \quad C_{GB} \equiv |S_{GB}|^2 = +2\,. \tag{21}$$

For an incident (negative-energy) Bernstein wave of energy $W_B^- = -1$, the transmitted Bernstein wave has energy $W_B^+ = T_B W_B^- = +1$, while the conversion to the gyroballistic mode has energy $W_G^+ = C_{GB} W_B^- = -2$. (Energy conservation thus reads $-1 = +1 - 2$.) As a result of wave propagation in (x, k_x) space, we note that once a wave leaves the coupling region there is no longer an opportunity for the waves to grow (since their interaction is now nonresonant).

We have shown that a Bernstein (collective) wave supported by an inverted minority-ion population can, in a nonuniform magnetic field, change the sign of its energy when it crosses the gyroresonance layer. It does so by transferring energy through linear mode conversion with a gyroballistic (noncollective) mode, which exists in the gyroresonance layer. Using a model distribution for the energetic minority ions, we obtained a set of two coupled equations near the gyroresonance layer, which we solved exactly. An exact energy conservation law was also derived. From the asymptotic properties of the solutions, we constructed an S-matrix relation between the eikonal amplitudes on both sides of the resonance layer. This relation was then used to obtain the explicit conversion and transmission coefficients: $C = 2$ and $T = -1$.

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