Variational structure for dissipationless linear drift-wave equations

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A derivation from first principles of the Lagrangian density for the dissipationless linear drift-wave equation recently introduced by Mattor and Diamond [Phys. Plasmas **1**, 4002 (1994)] is presented. An exact wave-action conservation law for linear drift waves propagating in a rotating magnetized plasma is then derived from the Lagrangian density without requiring the use of the eikonal representation for the wave fields. © *1996 American Institute of Physics.* [S1070-664X(96)04203-0]

I. INTRODUCTION

The variational structure for dissipationless dynamical equations relies on the existence of a Lagrangian density L.[1] By defining the action integral $A \equiv \int d^n r \; dt \; L$, where n is the number of spatial coordinates on which the variational fields depend, the dynamical equations are said to possess a variational structure if they are derivable from the variational principle $\delta A = 0$. The usefulness of this variational structure is that, through the Noether method,[2] each exact conservation law can be identified with an exact symmetry of the Lagrangian density L.

Recently, Mattor and Diamond,[3] in their investigation of the propagation of linear drift waves in a rotating tokamak plasma, introduced a variational principle for the dissipationless linear drift-wave equation. Their Lagrangian density was constructed from the linear drift-wave equation itself[4] and, except for some minor modifications, its generic form is given here as

$$L(\psi, d_t\psi, \nabla\psi, \nabla d_t\psi) \equiv \tfrac{1}{2}\,[a|d_t\psi|^2 + b|\nabla d_t\psi|^2 - (\mathbf{c}\cdot\nabla\psi)\,d_t\psi], \tag{1}$$

where the scalar field $\psi(\mathbf{x},t)$ is related to the perturbed electrostatic potential $\phi(\mathbf{x},t)$ according to

$$d_t\psi(\mathbf{x},t) = \phi(\mathbf{x},t). \tag{2}$$

In the simplest drift-wave model,[5] the wave field ψ is a function of the Cartesian coordinates $\mathbf{r}=(x,y)$ in the plane perpendicular to a *uniform* magnetic field $\mathbf{B}\equiv\hat{z}B$. In Eqs. (1) and (2), the total time derivative $d_t \equiv \partial_t + \mathbf{u}_0\cdot\nabla$ is taken along the (incompressible) *nonuniform* background flow velocity $\mathbf{u}_0 \equiv \hat{y}u_0(x)$, and the coefficients $(a,b,\mathbf{c})$ depend on the *nonuniform* background density $n_0(x)$. The coefficients a and b satisfy the conditions $d_t a = 0 = d_t b$, while the vector-valued coefficient $\mathbf{c}$ satisfies $\nabla\cdot\mathbf{c}=0$ and $[d_t, \mathbf{c}\cdot\nabla]=0$, i.e., their commutator vanishes.

Using the Lagrangian density (1), the variational principle $\delta(\int d^2r \; dt \; L)=0$ associated with arbitrary variations $\delta\psi(x,y,t)$ yields the Euler–Lagrange equation:

$$0 = \frac{\partial L}{\partial\psi} - \frac{d}{dt}\left(\frac{\partial L}{\partial(d_t\psi)}\right) - \nabla\cdot\left(\frac{\partial L}{\partial(\nabla\psi)}\right) + \frac{d}{dt}\,\nabla\cdot\left(\frac{\partial L}{\partial(\nabla d_t\psi)}\right),$$

or

$$0 = d_t[a \; d_t\psi - \nabla\cdot(b \; \nabla d_t\psi)] - \mathbf{c}\cdot\nabla d_t\psi, \tag{3}$$

where we used the fact that d_t commutes with $\mathbf{c}\cdot\nabla$. Upon substituting Eq. (2) into Eq. (3), we obtain a generic form of the linear drift-wave equation:[5]

$$0 = d_t[a\phi - \nabla\cdot(b \; \nabla\phi)] - \mathbf{c}\cdot\nabla\phi. \tag{4}$$

The purpose of this paper is to present a derivation of the Lagrangian density (1) from first principles, and derive explicit expressions for the coefficients $(a,b,\mathbf{c})$ in Eq. (4) for a magnetized plasma with a uniform magnetic field and nonuniform background density and background flow.

As a starting point for our derivation, we use the variational structure for the linearized drift-kinetic Vlasov–Poisson equations. This is motivated by the fact that the drift-wave equation (4), usually derived from two-fluid dissipationless equations,[5] can also be derived from the drift-kinetic Vlasov–Poisson equations by imposing[6] (a) the quasineutrality condition between the perturbed ion and electron densities, $n_{i1}=n_{e1}$; (b) the constraint that electrons have a Boltzmann distribution, $n_{e1}=n_0 e\phi/T_e$; and (c) the cold-fluid limit on the ion drift-kinetic Vlasov equation.

The drift-kinetic variational structure is derived in Sec. II from the variational structure for the linearized gyrokinetic Vlasov–Poisson equations (derived previously in Ref. 7). In the appropriate limit, the drift-wave variational structure is constructed from the drift-kinetic variational structure, and the Mattor–Diamond Lagrangian density (1) is recovered. In Sec. III, as an application of the drift-wave variational structure, an *exact* conservation law for wave action is derived from the drift-wave Lagrangian density through the Noether method,[2] without requiring the use of the eikonal representation for the wave fields. (Our derivation is thus different from the standard derivation,[8] which requires the use of the eikonal representation.[9]) We also comment on the wave-action conservation law for linear drift waves derived by Biskamp and Horton,[10] which was derived without the ben-

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efit of a Lagrangian formulation. Last, in Sec. IV we summarize our work and indicates several possible generalizations.

II. DERIVATION OF THE DRIFT-WAVE LAGRANGIAN DENSITY

A. Ion gyrokinetic variational structure

We derive the variational structure for the linearized drift-kinetic Vlasov–Poisson equations for a background magnetized plasma with a uniform magnetic field, a nonuniform background density profile $n_0(x)$, and a nonuniform background electrostatic potential $\phi_0(x)$. We begin with the *phase-space* Lagrangian density for the linearized ion gyrokinetic (gk) Vlasov equation in the presence of electrostatic fluctuations, which satisfy the gyrokinetic ordering ($\omega/\Omega_i \ll 1$, $k_\perp \rho_i \sim 1$). This Lagrangian density was shown in Ref. 7 to be a quadratic function of the perturbed electrostatic potential $\phi(\mathbf{r},t)$ and of the generating function $S_{\rm gk}(\mathbf{R},Z,U,\mu,t)$ for perturbations of the gyrokinetic Vlasov distribution:

$$F_1 \equiv \{S_{\rm gk}, F_0\}, \tag{5}$$

where $F_0(X,U,\mu)$ is the unperturbed guiding-center Vlasov distribution and $\{\,,\,\}$ is the guiding-center Poisson bracket. Here, $(\mathbf{R},Z,U,\mu,\theta)$ are guiding-center coordinates in a "local" frame moving with the background flow velocity $\mathbf{u}_0$ (see Appendix B of Ref. 11 for details): $\mathbf{R}=(X,Y)$ is the guiding-center position in the plane perpendicular to $\mathbf{B}$, Z is its position along $\mathbf{B}$, U is the parallel guiding-center velocity, μ is the magnetic moment, and θ is the gyroangle. (The dependence on the guiding-center coordinates Z and U is omitted in the remainder of this paper, since parallel gradients $\partial/\partial Z$ are assumed to vanish for all fields.)

Using Eq. (36) of Ref. 7 (with a vanishing perturbed magnetic field) and including nonuniform background flows, the ion gyrokinetic phase-space Lagrangian density is

$$\mathscr{L}_{\rm gk}^{(i)} = \{S_{\rm gk}, F_0\}\left(\frac{1}{2}\,\frac{dS_{\rm gk}}{dt} - e\langle\phi_{\rm gc}\rangle\right) + \frac{e^2F_0}{2\Omega_i}\,\langle\{\tilde{\Phi}_{\rm gc}, \tilde{\phi}_{\rm gc}\}\rangle, \tag{6}$$

where $\phi_{\rm gc}(\mathbf{R},\mu,\theta,t) \equiv \phi[\mathbf{r}(\mathbf{R},\mu,\theta),t]$ represents the perturbed electrostatic potential expressed in terms of guiding-center coordinates, with the particle position $\mathbf{r}(\mathbf{R},\mu,\theta) \equiv \mathbf{R} + \boldsymbol{\rho}(\mathbf{R},\mu,\theta)$ and $\boldsymbol{\rho}(\mathbf{R},\mu,\theta)$ is the gyroangle-dependent gyroradius, $\langle\phi_{\rm gc}\rangle$ is the gyroangle-averaged electrostatic potential, while $\tilde{\phi}_{\rm gc}$ and $\tilde{\Phi}_{\rm gc}$ are defined as $\tilde{\phi}_{\rm gc} \equiv \phi_{\rm gc} - \langle\phi_{\rm gc}\rangle$ and $\tilde{\Phi}_{\rm gc} \equiv \int \tilde{\phi}_{\rm gc}\, d\theta$. To describe guiding-center motion in a uniform magnetic field and a nonuniform background electrostatic potential ϕ_0, we need the guiding-center Hamiltonian,

$$H_0(X,U,\mu) = e\,\phi_0(X) + \mu B + m_i(U^2 + |\mathbf{u}_0|^2)/2, \tag{7}$$

where $\mathbf{u}_0(X) \equiv \hat{y}\,u_0(X)$ is the unperturbed $E\times B$ fluid velocity, with $u_0(X) = c\,\phi_0'(X)/B$, and the guiding-center Poisson bracket,

$$\{f,g\} = -\frac{c\hat{z}}{eB_\parallel^*}\cdot\boldsymbol{\nabla} f \times \boldsymbol{\nabla} g + \frac{\Omega_i}{B}\left(\frac{\partial f}{\partial\theta}\,\frac{\partial g}{\partial\mu} - \frac{\partial f}{\partial\mu}\,\frac{\partial g}{\partial\theta}\right), \tag{8}$$

where f and g are arbitrary functions of the guiding-center coordinates $(\mathbf{R},\mu,\theta)$, $\boldsymbol{\nabla} = \partial/\partial\mathbf{R}$ is the gradient perpendicular to $\mathbf{B}$, and

$$B_\parallel^* = B(1 + \Omega_i^{-1}\hat{z}\cdot\boldsymbol{\nabla}\times\mathbf{u}_0) \equiv \mathscr{T}B. \tag{9}$$

Using Eqs. (7)–(9), the unperturbed guiding-center Vlasov operator $d_t = \partial_t + \{\;\;, H_0\}$ is

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{u}_0\cdot\boldsymbol{\nabla}, \tag{10}$$

where $\boldsymbol{\nabla} H_0 = e\mathscr{T}\boldsymbol{\nabla}\phi_0$, with $\mathscr{T} = B_\parallel^*/B$ defined in Eq. (9), and we used $\mathbf{u}_0\cdot\boldsymbol{\nabla}\mathbf{u}_0 = 0$. In addition, to first order in background electrostatic potential nonuniformity,[11] the gyroradius vector is $\boldsymbol{\rho}(\mathbf{R},\mu,\theta) \equiv \boldsymbol{\rho}_0(\mu,\theta)/\mathscr{T}(X)$, where $\boldsymbol{\rho}_0$ is the gyroradius for an ion moving in a uniform background plasma $(|\boldsymbol{\rho}_0|^2 = \mu B/m_i\Omega_i^2)$.

The variational principle $\delta[\int d^2R\; dt(\int d\mu\; B_\parallel^*\mathscr{L}_{\rm gk}^{(i)})] = 0$ associated with arbitrary variations $\delta S_{\rm gk}(\mathbf{R},\mu,t)$ yields the linearized ion gyrokinetic Vlasov equation:

$$0 = \left[F_0, \left(\frac{dS_{\rm gk}}{dt} - e\langle\phi_{\rm gc}\rangle\right)\right] \equiv -\left(\frac{dF_1}{dt} + e\{F_0, \langle\phi_{\rm gc}\rangle\}\right). \tag{11}$$

As discussed in Appendix A of Ref. 7, the following evolution equation for $S_{\rm gk}$:

$$0 = \frac{dS_{\rm gk}}{dt} - e\langle\phi_{\rm gc}\rangle \tag{12}$$

is equivalent to Eq. (11). Since the operator d/dt in Eq. (12) is independent of μ [see Eq. (10)], the μ dependence of $S_{\rm gk}$ comes exclusively from $\langle\phi_{\rm gc}\rangle(\mathbf{R},\mu,t) \equiv \langle\phi(\mathbf{R}+\boldsymbol{\rho},t)\rangle$.

B. Ion drift-kinetic variational structure

The phase-space Lagrangian density (6) describes electrostatic fluctuations that satisfy the gyrokinetic ordering. A more appropriate ordering for drift-wave dynamics, however, is the drift-kinetic (DK) ordering ($\omega/\Omega_i \ll 1$, $k_\perp\rho_i \ll 1$). Hence, in the drift-kinetic limit, where $(\mu B/m_i\Omega_i^2)\nabla^2 \ll 1$; we find

$$\langle\phi_{\rm gc}\rangle \to \phi(\mathbf{R},t), \quad \tilde{\phi}_{\rm gc} \to \boldsymbol{\rho}\cdot\boldsymbol{\nabla}\phi(\mathbf{R},t), \quad \text{and}$$

$$\tilde{\Phi}_{\rm gc} \to \left(\int \boldsymbol{\rho}\; d\theta\right)\cdot\boldsymbol{\nabla}\phi(\mathbf{R},t). \tag{13}$$

When these expressions are substituted into the last term on the right side of Eq. (6), we obtain

$$\frac{e^2F_0}{2\Omega_i}\,\langle\{\tilde{\Phi}_{\rm gc}, \tilde{\phi}_{\rm gc}\}\rangle \to \frac{m_ic^2F_0}{2B_\parallel^{*2}}\,|\boldsymbol{\nabla}\phi|^2. \tag{14}$$

Next, applying the drift-kinetic limit (13) to the linearized ion gyrokinetic Vlasov equation (12), we obtain $dS_{\rm DK}/dt = e\,\phi(\mathbf{R},t)$, where $S_{\rm DK}(\mathbf{R},\mu,t)$ is the drift-kinetic limit of $S_{\rm gk}$. Since the evolution equation for $S_{\rm DK}$ is independent of μ, $S_{\rm DK}$ can be written as

$$S_{\rm DK}(\mathbf{R},\mu,t) \equiv \chi(\mathbf{R},t), \tag{15}$$

and χ satisfies the evolution equation $d\chi(\mathbf{R},t)/dt = e\,\phi(\mathbf{R},t)$.

The ion drift-kinetic Lagrangian density is thus obtained by substituting Eqs. (14) and (15) into Eq. (6):

$$\mathscr{L}_{\rm DK}^{(i)} = \frac{c\hat{z}}{eB_\parallel^*} \times \nabla F_0 \cdot \nabla \chi \left(\frac{1}{2} \frac{d\chi}{dt} - e\phi \right) + \frac{m_i c^2 F_0}{2B_\parallel^{*2}} |\nabla \phi|^2, \tag{16}$$

where $\{\chi, F_0\} = \nabla F_0 \times \nabla \chi \cdot c\hat{z}/eB_\parallel^*$ is the perturbed ion drift-kinetic Vlasov distribution or, equivalently, it is the gyroangle-averaged perturbed ion Vlasov distribution (in the drift-kinetic limit).

C. Drift-wave variational structure

In Eq. (16), we note that the only μ dependence resides in the unperturbed distribution $F_0(X,\mu)$. After integrating the phase-space Lagrangian density (16) over μ, with the background density defined as $n_0(X) \equiv \int d\mu\, B_\parallel^* F_0(X,\mu)$, we obtain the ion drift-wave (dw) Lagrangian density,

$$L_{\rm dw}^{(i)} \equiv \int d\mu\, B_\parallel^* \mathscr{L}_{\rm DK}^{(i)} = \frac{c\hat{z}}{eB} \times \nabla n_0^* \cdot \nabla \chi \left(\frac{1}{2} \frac{d\chi}{dt} - e\phi \right) + \frac{m_i c^2 n_0}{2B_\parallel^{*2}} |\nabla \phi|^2, \tag{17}$$

where $n_0^*(X) \equiv n_0(X)/\mathscr{T}(X)$ and $\int d\mu\, B_\parallel^* \nabla F_0 = \mathscr{T} \nabla n_0^*$. In the remainder of this paper, the wave fields χ and ϕ are functions of $\mathbf{r}=(x,y)$ and t, while n_0 and ϕ_0 are functions of x.

Next, we look at the Lagrangian density for an electron fluid with uniform temperature T_e. In the simplest drift-wave model,[5] the electrons are massless and have a Boltzmann distribution; their perturbed density is

$$n_{e1}(\mathbf{r},t) = n_0(x) e\phi(\mathbf{r},t)/T_e, \tag{18}$$

where T_e is the uniform electron temperature, and the Boltzmann-electron Lagrangian density is

$$L_{\rm dw}^{(e)} = \frac{n_0 e^2}{2T_e} \phi^2. \tag{19}$$

Last, we note that the electrostatic-field contribution $|\nabla\phi|^2/8\pi$ to the drift-wave Lagrangian density can be neglected, since it is smaller than the last term in Eq. (17) by a factor $c^2/v_{\rm A}^2 \gg 1$; neglecting it corresponds to imposing the quasineutrality condition between ions and electrons.

By combining the electron Lagrangian density (19) and the ion Lagrangian density (17), we obtain the Lagrangian density for the linear drift-wave dynamics:

$$L_{\rm dw} \equiv \frac{c\hat{z}}{eB} \times \nabla n_0^* \cdot \nabla \chi \left(\frac{1}{2} \frac{d\chi}{dt} - e\phi \right) + \frac{m_i c^2 n_0}{2B_\parallel^{*2}} |\nabla \phi|^2 + \frac{n_0 e^2}{2T_e} \phi^2. \tag{20}$$

The Lagrangian density (20) is a function of two *variational* fields $\phi(\mathbf{r},t)$ and $\chi(\mathbf{r},t)$. Requiring that $\int d^2 r\, dt\, L_{\rm dw}$ be stationary with respect to arbitrary variations in $\chi(\mathbf{r},t)$, we obtain

$$0 = \frac{c\hat{z}}{eB} \times \nabla n_0^* \cdot \nabla \left(\frac{d\chi}{dt} - e\phi \right), \tag{21}$$

which is the fluid moment of the linearized ion gyrokinetic Vlasov equation (11) in the drift-kinetic limit. Stationarity with respect to arbitrary variations in $\phi(\mathbf{r},t)$, on the other hand, yields

$$0 = -\frac{c\hat{z}}{B} \times \nabla n_0^* \cdot \nabla \chi - \nabla \cdot \left(\frac{m_i n_0 c^2}{B_\parallel^{*2}} \nabla \phi \right) + \frac{n_0 e^2}{T_e} \phi, \tag{22}$$

which represents the quasineutrality condition between ions and electrons.

Since the operators d/dt and $\hat{z} \times \nabla n_0^* \cdot \nabla$ commute, then applying the operator d/dt on Eq. (22), and substituting Eq. (21) yields

$$0 = \frac{d}{dt} \left[\frac{n_0 e^2}{T_e} \phi - \nabla \cdot \left(\frac{m_i n_0 c^2}{B_\parallel^{*2}} \nabla \phi \right) \right] - \frac{ce\hat{z}}{B} \times \nabla n_0^* \cdot \nabla \phi, \tag{23}$$

which describes linear drift waves propagating in a rotating magnetic plasma with a uniform magnetic field and a nonuniform background flow. In the absence of background flows (i.e., $\phi_0 = 0$, $B_\parallel^* = B$, and $n_0^* = n_0$), Eq. (23) is the standard linear drift-wave equation.[5] By comparing Eq. (23) with Eq. (4), the coefficients $(a,b,\mathbf{c})$ in Eq. (4) are thus

$$a = \frac{n_0 e^2}{T_e}, \quad b = \frac{m_i n_0 c^2}{B_\parallel^{*2}}, \quad \text{and} \quad \mathbf{c} = \frac{ce\hat{z}}{B} \times \nabla n_0^*. \tag{24}$$

Using (24), the drift-wave Lagrangian density (20) becomes

$$L_{\rm dw} = \mathbf{c} \cdot \nabla \psi \left(\frac{1}{2} \frac{d\psi}{dt} - \phi \right) + \frac{1}{2} (b|\nabla \phi|^2 + a\phi^2), \tag{25}$$

where $\psi(\mathbf{r},t) \equiv \chi(\mathbf{r},t)/e$. If we substitute $\phi = d_i \psi$ into Eq. (25), the Mattor–Diamond Lagrangian density (1) is recovered.

We conclude this section by noting that the quasineutrality condition (22) yields the following expression for the perturbed ion density:

$$n_{i1} = -\nabla \cdot \left[n_0 \left(\frac{c\hat{z}}{eB_\parallel^*} \times \nabla \chi - \frac{m_i c^2}{eB_\parallel^{*2}} \nabla \phi \right) \right]. \tag{26}$$

The first term on the right side of Eq. (26) corresponds to the fluid moment of the perturbed ion drift-kinetic Vlasov distribution,[6] $\int d\mu\, B_\parallel^* \{\chi, F_0\}$, while the second term corresponds to the ion polarization density.[5] Furthermore, by solving the perturbed cold-ion fluid equation of motion,

$$d_t \mathbf{u}_{i1} + \mathbf{u}_{i1} \cdot \nabla \mathbf{u}_0 = \Omega_i (\mathbf{u}_{i1} \times \hat{z} - c\nabla\phi/B) \tag{27}$$

to first order in Ω_i^{-1}, we obtain the following expression for the perturbed ion fluid velocity:

$$\mathbf{u}_{i1} = \frac{c\hat{z}}{B_\parallel^*} \times \nabla \phi - \frac{m_i c^2}{eB_\parallel^{*2}} (d_t \nabla \phi - \nabla \phi \cdot \nabla \mathbf{u}_0). \tag{28}$$

Thus, we write the linear drift-wave equation (23) as the perturbed ion continuity equation, $d_t n_{i1} = -\nabla \cdot (n_0 \mathbf{u}_{i1})$, with the perturbed ion density $n_{i1} = n_{e1} = n_0 e\phi/T_e$ (making use of the quasineutrality condition and the constraint that electrons have a Boltzmann distribution) and the perturbed ion fluid velocity (28).

Last, by comparing Eqs. (26) and (27) with the Eulerian expressions[12] for the perturbed density $n_{i1} \equiv -\nabla \cdot (n_0 \boldsymbol{\xi}_i)$ and for the perturbed fluid velocity $\mathbf{u}_{i1} \equiv d_t \boldsymbol{\xi}_i - \boldsymbol{\xi}_i \cdot \nabla \mathbf{u}_0$ in terms of the ion fluid displacement $\boldsymbol{\xi}_i$, we find the following expression for $\boldsymbol{\xi}_i$:

$$\boldsymbol{\xi}_i = \frac{c\hat{z}}{eB_\parallel^*} \times \nabla \chi - \frac{m_i c^2}{eB_\parallel^{*2}} \nabla \phi. \tag{29}$$

This expression has a simple interpretation in terms of the particle displacement $\delta \mathbf{r}(\mathbf{r},\mathbf{v}) \equiv \{\mathbf{r}, S(\mathbf{r},\mathbf{v})\}_p$, where S is the generating function for a canonical transformation on particle (p) phase space and $\{\ ,\ \}_p$ is the Poisson bracket (for simplicity of presentation, the time dependence is omitted). Writing $\delta \mathbf{r}$ in the guiding-center representation, we find $\delta \mathbf{r}(\mathbf{R},\mu,\theta) = [\mathbf{R} + \boldsymbol{\rho},\ S_{\rm gc}(\mathbf{R},\mu,\theta)]$, where $S_{\rm gc} \equiv S_{\rm gk} + e\tilde{\Phi}_{\rm gc}/\Omega_i$ is the generating function in the guiding-center representation and $\{\ ,\ \}$ is given by Eq. (8) (see Ref. 7 for details on the decomposition of $S_{\rm gc}$ used here). If we now gyroangle average $\delta \mathbf{r}(\mathbf{R},\mu,\theta)$, the terms $\langle \{\boldsymbol{\rho},\ S_{\rm gk}\} \rangle$ and $\langle \{\mathbf{R}, e\tilde{\Phi}_{\rm gc}/\Omega_i\} \rangle$ vanish, and we are left with

$$\langle \delta\ \mathbf{r} \rangle(\mathbf{R},\mu) = \{\mathbf{R}, S_{\rm gk}\} - \frac{e}{B} \frac{\partial}{\partial \mu} \langle \boldsymbol{\rho} \tilde{\phi}_{\rm gc} \rangle. \tag{30}$$

In the drift-kinetic limit [see Eqs. (13) and (15)], the gyroangle-averaged particle displacement (30) becomes $\langle \delta\ \mathbf{r} \rangle = \{\mathbf{R}, \chi\} - (m_i c^2/eB_\parallel^{*2}) \nabla \phi$, which is exactly equal to the fluid displacement (29). Hence, the ion fluid displacement (29) is composed of (1) the particle displacement generated by the drift-kinetic generating function χ and (2) the ion-polarization displacement.

III. WAVE-ACTION CONSERVATION FOR LINEAR DRIFT WAVES

In this section, an application of the variational structure for the linear drift-wave equation, based on the drift-wave Lagrangian density (25), is presented. Through the Noether method,[2] we derive an exact wave-action conservation law for drift waves propagating in a magnetized plasma with nonuniform background flows. This conservation law, derived following a procedure presented elsewhere,[13,14] has the form

$$\frac{\partial J_{\rm dw}}{\partial t} + \nabla \cdot \boldsymbol{\Gamma}_{\rm dw} = 0, \tag{31}$$

where $J_{\rm dw}(\mathbf{r},t)$ is the wave-action density and $\boldsymbol{\Gamma}_{\rm dw}(\mathbf{r},t)$ is the wave-action-density flux.

As a first step in deriving the wave-action conservation law (31), the drift-wave Lagrangian density (25) is transformed into the *averaged* drift-wave Lagrangian density:

$$\begin{aligned} \bar{L}_{\rm dw} \equiv& \mathbf{c} \cdot [\tfrac{1}{2}(d_i \psi \nabla \psi^* + d_t \psi^*\ \nabla \psi) - (\phi\ \nabla \psi^* + \phi^*\ \nabla \psi)] \\ &+ a|\phi|^2 + b|\nabla \phi|^2, \end{aligned} \tag{32}$$

where both ϕ and ψ are now complex-valued scalar fields. The wave-action conservation law (31) follows from the fact that the drift-wave Lagrangian density (32) is invariant under the phase shift: $\phi \rightarrow \phi e^{i\alpha}$ and $\psi \rightarrow \psi e^{i\alpha}$, where the constant phase α is real. The wave-action density $J_{\rm dw}$ and wave-action-density flux $\Gamma_{\rm dw}$ in Eq. (31) are defined, respectively, in terms of the averaged Lagrangian density (32) as[14]

$$\begin{aligned} J_{\rm dw} &\equiv 2\ \mathrm{Im}\left(\psi \frac{\partial \bar{L}_{\rm dw}}{\partial(\partial_t \psi)} \right) + 2\ \mathrm{Im}\left(\phi \frac{\partial \bar{L}_{\rm dw}}{\partial(\partial_t \phi)} \right), \\ \Gamma_{\rm dw} &\equiv 2\ \mathrm{Im}\left(\psi \frac{\partial \bar{L}_{\rm dw}}{\partial(\nabla \psi)} \right) + 2\ \mathrm{Im}\left(\phi \frac{\partial \bar{L}_{\rm dw}}{\partial(\nabla \phi)} \right). \end{aligned} \tag{33}$$

By substituting the averaged Lagrangian density (32) into Eq. (33), we thus find

$$\begin{aligned} J_{\rm dw} &= \mathrm{Im}(\psi \mathbf{c} \cdot \nabla \psi^*), \\ \boldsymbol{\Gamma}_{\rm dw} &= \mathbf{u}_0 J_{\rm dw} - \mathbf{c}\ \mathrm{Im}\left(\psi \frac{d\psi^*}{dt} \right) + 2b\ \mathrm{Im}(\phi\ \nabla \phi^*). \end{aligned} \tag{34}$$

One can check that (34) satisfy the wave-action conservation law (31) exactly, using Eqs. (21) and (22), without requiring the use of the standard eikonal representation for the wave fields ϕ and ψ.

We now compare the exact expressions (34) for $J_{\rm dw}$ and $\boldsymbol{\Gamma}_{\rm dw}$ with the standard expressions obtained in eikonal wave theory.[8] Using the eikonal representation for the wave fields ψ and ϕ,

$$\begin{pmatrix} \psi(x,y,t) \\ \phi(x,y,t) \end{pmatrix} \equiv \begin{pmatrix} \hat{\psi}(x,y,t) \\ \hat{\phi}(x,y,t) \end{pmatrix} e^{i\Theta(x,y,t)}, \tag{35}$$

where $\Theta(x,y,t)$ is the eikonal phase, with the local wave vector $\mathbf{k} \equiv \nabla \Theta$ and the local wave frequency $\omega \equiv -\partial_t \Theta$, and $(\hat{\psi}, \hat{\phi})$ are slowly varying eikonal amplitudes, the drift-wave Lagrangian density (25) becomes

$$\begin{aligned} \hat{L}_{\rm dw} &= -\omega' \mathbf{k} \cdot \mathbf{c} |\hat{\psi}|^2 + i\mathbf{k} \cdot \mathbf{c}(\hat{\phi}\hat{\psi}^* - \hat{\phi}^* \hat{\psi}) + (a + bk^2)|\hat{\phi}|^2 \\ &= \omega'[\mathbf{k} \cdot \mathbf{c} + \omega'(a + b|\mathbf{k}|^2)]|\hat{\psi}|^2, \end{aligned} \tag{36}$$

where $\omega' \equiv \omega - \mathbf{k} \cdot \mathbf{u}_0$ and $\hat{\phi} = -i\omega' \hat{\psi}$ was used to obtain the second equation. The dispersion relation for linear drift waves,

$$\omega = \mathbf{k} \cdot \mathbf{u}_0 - \mathbf{k} \cdot \mathbf{c}/(a + b|\mathbf{k}|^2), \tag{37}$$

is obtained from $\partial \hat{L}_{\rm dw}/\partial \hat{\psi} = 0$ and $\omega' \neq 0$, $\hat{\psi} \neq 0$. If we now apply the eikonal representation (35) on (34), we obtain the standard results:[8]

$$J_{\rm dw} \equiv \frac{\partial \hat{L}_{\rm dw}}{\partial \omega} \quad \text{and} \quad \boldsymbol{\Gamma}_{\rm dw} \equiv -\frac{\partial \hat{L}_{\rm dw}}{\partial \mathbf{k}} = \left(\frac{\partial \omega}{\partial \mathbf{k}} \right) J_{\rm dw}, \tag{38}$$

where Eqs. (36)–(37) were used and the group velocity $\partial\omega/\partial\mathbf{k}$ is calculated from the dispersion relation (37).

The energy density in the *lab* frame, denoted $E_{\rm dw}$, is calculated from the *eikonal*-averaged Lagrangian density (36) as

$$E_{\rm dw} \equiv \omega \frac{\partial \hat{L}_{\rm dw}}{\partial \omega} = -\omega \mathbf{k} \cdot \mathbf{c} |\hat{\psi}|^2, \tag{39}$$

while the energy density in the *moving* frame, denoted $E'_{\rm dw}$, is

$$E'_{\rm dw} \equiv \omega' \frac{\partial \hat{L}_{\rm dw}}{\partial \omega'} = (a + b|\mathbf{k}|^2)|\hat{\phi}|^2, \tag{40}$$

where $\hat{\phi} = -i\omega' \hat{\psi}$ was used. (The relation $E'_{\rm dw} = E_{\rm dw} + \mathbf{P}_{\rm dw} \cdot \mathbf{u}_0$ between these two energy densities involves the drift-wave momentum density, $\mathbf{P}_{\rm dw} \equiv -\mathbf{k}\ \partial \hat{L}_{\rm dw}/\partial\omega$.) We can check that, in the eikonal limit, the drift-wave action density is expressed either as $J_{\rm dw}=E_{\rm dw}/\omega$ or as $J_{\rm dw}=E'_{\rm dw}/\omega'$ since, for an arbitrary eikonal-averaged Lagrangian $\hat{L}$ and an arbitrary unperturbed flow velocity $\mathbf{u}$, we have the identity

$$\frac{E'}{\omega'}=\frac{E+\mathbf{P}\cdot\mathbf{u}}{\omega-\mathbf{k}\cdot\mathbf{u}}=\frac{E+(-\mathbf{k}E/\omega)\cdot\mathbf{u}}{\omega-\mathbf{k}\cdot\mathbf{u}}=\frac{E(1-\mathbf{k}\cdot\mathbf{u}/\omega)}{\omega-\mathbf{k}\cdot\mathbf{u}}=\frac{E}{\omega}.$$

Hence, although the wave frequency ω and the wave energy $E_{\rm dw}$ are not Galilean invariant, the wave-action density is Galilean invariant. We note that the property that the wave-action density be equal to the ratio of the wave energy over the wave frequency is a universal property of all eikonal-averaged Lagrangians.

Finally, we comment on the wave-action conservation law for linear drift waves in a stationary magnetized plasma previously derived by Biskamp and Horton.[10] Since the variational structure for linear drift waves was not known at that time, Biskamp and Horton derived a conservation law directly from the linear drift-wave equation; because the conserved quantity that they obtained was proportional to the drift-wave energy $(|\phi|^2+|\nabla\phi|^2)$, Biskamp and Horton concluded that, contrary to all other linear dissipationless wave models (e.g., linear waves propagating on a string), the wave-action density for linear drift waves was equal to its wave-energy density. The flaw in their construction, in our opinion, is that the background density was omitted from their expression for the wave-energy density [see Eq. (7) of Ref. 10]. For a time-*independent* background plasma, the conservation laws of wave energy and wave action are quite familiar; however, for a time-*dependent* background plasma, the wave energy is no longer conserved (i.e., energy can be transferred between the waves and the background plasma) while the wave action is still conserved (i.e., the number of wave quanta is constant). Hence, as expected, the conservation law derived by Biskamp and Horton appears to be a wave-*energy* conservation law (not a wave-*action* conservation law), and this law is valid only for a time-independent background plasma. The first correct derivation of the wave-action conservation law for linear drift waves in two dimensions,[15] within the standard eikonal representation,[8] appears to have been carried out by Mattor and Diamond[3] with the help of the Lagrangian density (1).

IV. SUMMARY

A derivation from first principles of the Mattor–Diamond Lagrangian density (1) for linear drift waves in a rotating magnetized plasma was presented. The drift-wave Lagrangian density (20), from which the Lagrangian density (1) can be derived, is a function of two variational fields: the perturbed electrostatic potential ϕ and the generating function χ for the perturbed Vlasov distribution function (in the drift-kinetic limit). We showed, in Sec. II, how the Lagrangian density (20) was derived from the phase-space Lagrangian densities for the linearized drift-kinetic and gyrokinetic Vlasov–Poisson equations. In Sec. III, we derived an exact wave-action conservation law associated with linear drift waves propagating in a rotating magnetized plasma.

The usefulness of our derivation can be made apparent when we consider generalizations of the linear drift-wave equation (23). In particular, more realistic background magnetized plasmas can be easily accommodated through the drift-kinetic and/or gyrokinetic variational structures. For example, nonuniform background magnetic fields, ion kinetic, or thermal fluid effects, and nonadiabatic electron dynamics can be included into the drift-wave Lagrangian density (20). These additional effects, as well as the investigation of the variational structure for the dissipationless nonlinear drift-wave equations (e.g., the Hasegawa–Mima equation[5]) will be treated in a future work.

Finally, we point out that the present variational formalism based on the Lagrangian density (1) can also be applied to the linearized Rossby-wave equation.[5]

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[1] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953), Part I, Chap. 3.

[2] P. J. Olver, *Applications of Lie Groups to Differential Equations* (Springer-Verlag, New York, 1986), Chap. 4.

[3] N. Mattor and P. H. Diamond, Phys. Plasmas **1**, 4002 (1994), see Appendix B.

[4] N. Mattor (private communication, 1995).

[5] A. Hasegawa, C. G. Maclennan, and Y. Kodama, Phys. Fluids **22**, 2122 (1979); for a recent review, see W. Horton and A. Hasegawa, Chaos **4**, 227 (1994).

[6] E. A. Frieman and L. Chen, Phys. Fluids **25**, 502 (1982), Sec. III B; D. H. E. Dubin, J. A. Krommes, C. Oberman, and W. W. Lee, Phys. Fluids **26**, 3524 (1983), Sec. V.

[7] A. Brizard, Phys. Plasmas **1**, 2460 (1994).

[8] G. B. Whitham, *Linear and Nonlinear Waves* (Wiley, New York, 1974), Sec. 7.7.

[9] Within the eikonal representation (Ref. 8), the standard wave-action conservation law is only asymptotically valid (i.e., in the limit where the background medium is weakly nonuniform and weakly time dependent compared to the wave space–time scales) and wave action is considered as an adiabatic invariant; the wave-action conservation law derived in this paper, which is valid for noneikonal wave fields (i.e., wave fields for which the eikonal approximation breaks down), is, therefore, a generalization of the standard wave-action conservation law.

[10] D. Biskamp and W. Horton, Phys. Lett. A **75**, 359 (1980).

[11] A. J. Brizard, Phys. Plasmas **2**, 459 (1995).

[12] W. A. Newcomb, Nucl. Fusion, 1962 Suppl. (Part 2), 451 (1962).

[13] A. J. Brizard, D. R. Cook, and A. N. Kaufman, Phys. Rev. Lett. **70**, 521 (1993).

[14] A. J. Brizard and A. N. Kaufman, Phys. Rev. Lett. **74**, 4567 (1994).

[15] A. Lagrangian density for a one-dimensional model for drift waves was introduced by P. J. Morrison, J. D. Meiss, and J. R. Cary [Physica D **11**, 324 (1984)], from which a wave-action conservation law for one-dimensional linear drift waves can be derived.