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Wave-Action Conservation for Pseudo-Hermitian Fields

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The linearized Vlasov-Maxwell system and ideal magnetohydrodynamics are examples of systems with a *pseudo-Hermitian* structure with respect to an *indefinite* inner product on the space of complex representations of real fields [J. Larsson, Phys. Rev. Lett. **66**, 1466 (1991)]. This structure leads to variational principles, whose variation with respect to local wave phase yields new wave-action conservation laws, which generalize previous eikonal versions.

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For some years, the law of wave-action conservation for classical fields has been based on the *eikonal* approximation, which requires that wave amplitudes are slowly varying spatially on the scale of the wave phase [1–3]. Yet there are important situations in physics where this approximation is clearly invalid, and where intuition leads one still to expect action conservation. The need for such a conservation law is especially important for wave propagation in a medium with weak time dependence, where wave-energy conservation is lost. Some common situations where eikonality fails are the following: (1) Waves propagate across *discontinuities*, undergoing refraction and partial reflection; (2) in plasma kinetic models, wave action is transferred between collective and kinetic modes, often over a *singular layer* [4–6]; and (3) waves are *scattered* as they propagate through a turbulent medium [7].

Recently Kull, Berk, and Morrison [8] presented a noneikonal formulation of wave-action conservation for a field whose evolution is governed by a *Hermitian* operator. This requirement of Hermiticity restricts its applicability by omitting effects such as wave-particle resonance in Vlasov theory. However, there is an important class of systems whose wave operators are *pseudo-Hermitian* [9], in the sense that Hermiticity is defined with respect to an *indefinite-metric* inner product [10].

Several important dynamical systems have a *Hamiltonian* structure in terms of a generalized (noncanonical) Poisson bracket [11]. The best-known examples in plasma physics are ideal magnetohydrodynamics (MHD)

[12] and the Vlasov-Maxwell model of collisionless plasma dynamics [13]. In the process of linearization, the first-order fields represent perturbations from an arbitrary (possibly time-dependent) *reference* state. These fields can be expressed in terms of generating functions; in linearized ideal MHD theory, for example, the linear fields ($\rho_1, \mathbf{u}_1, p_1, \mathbf{B}_1$) are expressed in terms of the fluid displacement $\boldsymbol{\xi}(\mathbf{x}, t)$ [e.g., $\rho_1 = -\nabla \cdot (\rho_0 \boldsymbol{\xi})$].

For the Vlasov-Maxwell system, Larsson [9] developed the linearization in terms of $S(z; t)$, a scalar generating function for canonical transformations on six-dimensional phase space $z \equiv (\mathbf{r}, \mathbf{p} \equiv m\mathbf{v})$. The linear perturbation of the Vlasov distribution, f_1 , is generated jointly by S and $\mathbf{A}_1(\mathbf{x}, t)$ (the linearized vector potential) acting on the reference distribution $f_0(z; t)$: $f_1 = \{S, f_0\} + (e/c)\mathbf{A}_1(\mathbf{r}, t) \cdot \partial f_0 / \partial \mathbf{p}$. This represents dynamically accessible perturbations [14–16]. Larsson showed that, by *complexifying* the linear fields, and by defining an indefinite-metric inner product (with zero norm for real fields), the time-evolution operator was indeed Hermitian with respect to this unconventional inner product. Upon our recognition that this pseudo-Hermiticity was a general property of linearized Hamiltonian systems [17], one of us developed the appropriate formulation for linearized MHD [18].

An immediate consequence of the pseudo-Hermitian formulation is its associated quadratic *variational principle*. In this paper we show how to use this principle to construct the wave-action continuity equation, by considering an arbitrary *nonuniform* phase shift of the complex-

ified wave field. The three examples we treat are the following: (1) ideal MHD, of which fluid dynamics is a special case; (2) cold multifluid electrodynamics, which also has a noncanonical Hamiltonian structure; and (3) the Vlasov-Maxwell system, of which the Vlasov-Coulomb system is a special case. In each case, the reference state (about which linearization is taken) is allowed to be time dependent.

The expressions obtained for the wave-action density and flux are local quadratic functions of the *complexified* wave fields [see, e.g., Eqs. (4), (8), and (15)]. Complex representations have been common for real linear fields, but have usually been considered merely an algebraic convenience. Here, however, complexification is *essential*, as these expressions vanish identically for real fields. [One accepted algorithm for complexification is Gabor's analytic signal [19]: For any real function of time $f(t)$, define the analytic $f_A(t) \equiv \int_0^\infty (d\omega/\pi) e^{-i\omega t} \bar{f}(\omega)$, with $\bar{f}(\omega) \equiv \int_{-\infty}^{+\infty} dt e^{+i\omega t} f(t)$. The inverse is simply $f(t) = \text{Re } f_A(t)$.]

Our derivation of the wave-action continuity equation

$$\partial_t J(\mathbf{x}; t) + \nabla \cdot \mathbf{J}(\mathbf{x}; t) = 0, \quad (1)$$

for the action density J and the action flux density \mathbf{J} , is based on a variational functional \mathcal{A} for the complexified wave field. Because \mathcal{A} is stationary for *all* infinitesimal variations of the field about a solution, it is stationary for the set of restricted variations representing infinitesimal *nonuniform* phase shifts. This is the generalization of the space-time variation of the eikonal phase in the standard derivations for eikonal wave fields. Formally, (1) states that the functional derivative of \mathcal{A} with respect to the wave phase vanishes.

For each of the three examples presented in this paper, we have explicitly evaluated $\partial J/\partial t$ by using (4), (8), or (15), the evolution equations for an *arbitrary* reference state, and the evolution equations for the wave field. In each case, we have rigorously verified that Eq. (1) holds with \mathbf{J} given by (5), (9), or (16), respectively, with *no* approximation.

We illustrate the method by considering first the case of ideal single-fluid MHD, where the wave field is represented by the fluid displacement $\xi(\mathbf{x}, t)$. The variational functional, yielding the standard evolution equation for ξ [but allowing for *time-dependent* reference state $\mathbf{B}_0(\mathbf{x}, t)$, $\rho_0(\mathbf{x}, t)$, $\mathbf{u}_0(\mathbf{x}, t)$, $p_0(\mathbf{x}, t)$], is [18, 20]

$$\mathcal{A}(\xi) \equiv \frac{1}{2} \int d^4x \{ \rho_0 |\dot{\xi}|^2 + \xi^* \cdot [\nabla \cdot \mathbf{K}(\xi)] \}, \quad (2)$$

in terms of the Hermitian operator $\nabla \cdot \mathbf{K}$, where

$$\mathbf{K}(\xi) \equiv -(p_1 + \mathbf{B}_0 \cdot \mathbf{Q}) \mathbf{1} + \mathbf{B}_0 \mathbf{Q} + \mathbf{Q} \mathbf{B}_0 + \xi \rho_0 \dot{\mathbf{u}}_0,$$

$\mathbf{Q}(\xi) \equiv \nabla \times (\xi \times \mathbf{B}_0)/4\pi \equiv \mathbf{B}_1/4\pi$, $p_1(\xi) \equiv -\xi \cdot \nabla p_0 - \gamma p_0 \nabla \cdot \xi$, $\dot{\xi} \equiv \partial \xi / \partial t + \mathbf{u}_0 \cdot \nabla \xi$. In (2), we take the variation of \mathcal{A} for $\xi(\mathbf{x}, t) \rightarrow \xi(\mathbf{x}, t) \exp i\alpha(\mathbf{x}, t)$, with α infinitesimal and arbitrary; thus $\delta \xi = i\alpha \xi$, $\delta \xi^* = -i\alpha \xi^*$. After some integration by parts (letting α have compact support), and using the Hermiticity of \mathbf{K} , we find

$$\delta \mathcal{A} = \int d^4x \alpha(\mathbf{x}, t) [\partial J / \partial t + \nabla \cdot \mathbf{J}], \quad (3)$$

with the action density

$$J(\mathbf{x}; t) \equiv \text{Im} [\rho_0(\mathbf{x}, t) \dot{\xi}^*(\mathbf{x}, t) \cdot \xi(\mathbf{x}, t)], \quad (4)$$

and the action flux density

$$\mathbf{J}(\mathbf{x}; t) \equiv \mathbf{J}^{(0)}(\mathbf{x}; t) + \mathbf{u}_0(\mathbf{x}, t) J(\mathbf{x}; t), \quad (5)$$

where

$$\mathbf{J}^{(0)}(\mathbf{x}; t) \equiv \text{Im} [(\xi^* \times \mathbf{B}_0) \times \mathbf{Q} - \xi^* p_1 - (\xi^* \times \xi \cdot \mathbf{j}_0) \mathbf{B}_0 / 2c] \quad (6)$$

is the action flux density in the local rest frame, and $\mathbf{u}_0 J$ is the convective contribution. Because $\delta \mathcal{A} = 0$ for all α , we obtain the desired action continuity equation (1).

The expressions (4) and (5) have the following properties: (a) For real ξ , they vanish *identically*; (b) because $\dot{\xi}$ is the *convective* time derivative, J is invariant under Galilean frame changes; and (c) in the *eikonal* limit, $\xi \rightarrow -i(\omega - \mathbf{k} \cdot \mathbf{u}_0) \xi$, $J(\mathbf{x}, t) \rightarrow (\omega - \mathbf{k} \cdot \mathbf{u}_0) \rho_0 |\xi|^2$, and $\mathbf{J} = J \times (\text{local group velocity})$, the standard results in MHD [2].

When the magnetic terms are deleted from the MHD formulas, one obtains results appropriate for fluid dynamics. They can also be generalized in several directions: to rotating frames [21], to dielectric fluids, and to special-relativistic [22, 23] and general-relativistic [24] fluids.

We next consider ideal cold multispecies electrodynamics [25]. The first-order fields are $(\xi_s, \phi_1, \mathbf{A}_1)$, where ξ_s denotes the displacement for species s , and (ϕ_1, \mathbf{A}_1) are the first-order electromagnetic potentials. The reference state is represented by the (time-dependent) fields: n_0^s , \mathbf{u}_0^s , \mathbf{E}_0 , \mathbf{B}_0 . The variational functional can be derived from the cold-fluid limit of the Low Lagrangian [26]:

$$\begin{aligned} \mathcal{A}(\phi_1, \mathbf{A}_1, \xi_s) \equiv & \int d^4x \left[\frac{|\mathbf{E}_1|^2 - |\mathbf{B}_1|^2}{8\pi} + \sum_s \frac{n_0^s m_s}{2} |\dot{\xi}_s|^2 \right] \\ & + \text{Re} \int d^4x \sum_s n_0^s e_s \{ \xi_s^* \times \xi_s \cdot \mathbf{B}_0 / 2c + \frac{1}{2} \xi_s^* \cdot [\nabla \mathbf{E}_0 - (\nabla \mathbf{B}_0) \times \mathbf{u}_0^s / c] \cdot \xi_s \\ & + \xi_s^* \cdot (\mathbf{E}_1 + \mathbf{u}_0^s \times \mathbf{B}_1 / c) \}, \end{aligned} \quad (7)$$

with $\mathbf{E}_1 \equiv -\nabla\phi_1 - c^{-1}\partial\mathbf{A}_1/\partial t$, and $\mathbf{B}_1 \equiv \nabla \times \mathbf{A}_1$. By the same method as for MHD, using the same α for ϕ_1 , \mathbf{A}_1 , ξ_s ($\delta\phi_1 = i\alpha\phi_1$, $\delta\mathbf{A}_1 = i\alpha\mathbf{A}_1$, $\delta\xi_s = i\alpha\xi_s$), we derive the action density

$$J(\mathbf{x}; t) \equiv \text{Im} \left\{ \mathbf{A}_1^* \cdot \mathbf{E}_1/4\pi c + \sum_s \frac{n_0^s e_s}{2c} \mathbf{B}_0 \cdot \xi_s^* \times \xi_s - \sum_s n_0^s \xi_s^* \cdot \left(m_s \dot{\xi}_s + \frac{e_s}{c} \mathbf{A}_1 \right) \right\} \quad (8)$$

and the action flux density

$$\mathbf{J}(\mathbf{x}; t) \equiv \text{Im} \left((\phi_1^* \mathbf{E}_1 + \mathbf{A}_1^* \times \mathbf{B}_1)/4\pi + \sum_s n_0^s \left\{ \mathbf{u}_0^s \left[-\xi_s^* \cdot \left(m_s \dot{\xi}_s + \frac{e_s}{2c} \mathbf{A}_1 \right) + \frac{e_s}{2c} \mathbf{B}_0 \cdot \xi_s^* \times \xi_s \right] - e_s \xi_s^* \left(\phi_1 - \frac{\mathbf{u}_0^s}{c} \cdot \mathbf{A}_1 \right) \right\} \right). \quad (9)$$

In the *eikonal* limit $\mathbf{A}_1 \equiv \tilde{\mathbf{A}}(\mathbf{x}, t) \exp i\Theta(\mathbf{x}, t)$, the species displacements $\xi_s(\mathbf{x}, t)$ can be algebraically expressed in terms of the local linear electromagnetic field. Then the action density reduces (in the radiation gauge $\phi_1 \equiv 0$) to the standard expression [3]

$$J = \tilde{\mathbf{A}}^* \cdot (\partial D/\partial \omega) \cdot \tilde{\mathbf{A}}, \quad (10)$$

and the action flux density to

$$\mathbf{J} = -\tilde{\mathbf{A}}^* \cdot (\partial D/\partial \mathbf{k}) \cdot \tilde{\mathbf{A}}, \quad (11)$$

where $D(\mathbf{k}, \omega; \mathbf{x}, t)$ is the local *cold-plasma* dispersion tensor (allowing for nonzero flow \mathbf{u}_0^s). On introducing the local polarization by $\tilde{\mathbf{A}} = \hat{\mathbf{e}}(\mathbf{x}, t) A(\mathbf{x}, t)$, these reduce further to the standard forms

$$J = (\partial D/\partial \omega) |A|^2, \quad (12)$$

$$\mathbf{J} = J \times (\text{local group velocity}), \quad (13)$$

where $D \equiv \hat{\mathbf{e}}^* \cdot \mathbf{D} \cdot \hat{\mathbf{e}}$. Also [3], in the eikonal limit, the wave-energy density is ωJ , and the wave-momentum density is $\mathbf{k}J$.

Finally we turn to the Vlasov-Maxwell model of collisionless plasma dynamics. Larsson [27] has recently derived the variational functional for the Vlasov-Coulomb model from the Low Lagrangian [26]. The generalization to the Vlasov-Maxwell model is straightforward; the first-order fields are the generating function $S(z; t)$ and the electromagnetic potentials $\phi_1(x)$ and $\mathbf{A}_1(x)$. The variational functional is

$$\mathcal{A}(\mathbf{A}_1, \phi_1, S) \equiv \int d^4x \left(|\mathbf{E}_1|^2 - |\mathbf{B}_1|^2 \right) / 8\pi - \frac{1}{2} \int dt \int d^6z f_0(z; t) \left[e^2 |\mathbf{A}_1|^2 / mc^2 + \text{Re} \{ S^*, \dot{S} - 2e(\phi_1 - \mathbf{v} \cdot \mathbf{A}_1/c) \} \right], \quad (14)$$

where

$$\dot{S} \equiv \frac{\partial S}{\partial t} + \{S, h_0\} - \frac{e}{c} \frac{\partial \mathbf{A}_0}{\partial t} \cdot \frac{\partial S}{\partial \mathbf{p}},$$

and

$$h_0(z, t) \equiv p^2/2m + e\phi_0(\mathbf{r}, t),$$

summation over species is implicit, and the noncanonical Poisson bracket [28] is

$$\{f, g\} \equiv \frac{\partial f}{\partial \mathbf{r}} \cdot \frac{\partial g}{\partial \mathbf{p}} - \frac{\partial f}{\partial \mathbf{p}} \cdot \frac{\partial g}{\partial \mathbf{r}} + \frac{e}{c} \mathbf{B}_0(\mathbf{r}, t) \cdot \frac{\partial f}{\partial \mathbf{p}} \times \frac{\partial g}{\partial \mathbf{p}},$$

in terms of the possibly time-dependent *reference* field \mathbf{B}_0 .

We proceed as before and obtain for the action density

$$J(\mathbf{x}; t) \equiv \text{Im} \left[\mathbf{A}_1^* \cdot \mathbf{E}_1/4\pi c + \frac{1}{2} \int d^6z \delta^3(\mathbf{x} - \mathbf{r}) \{S^*, f_0\} S \right], \quad (15)$$

and the action flux density is

$$\mathbf{J}(\mathbf{x}; t) \equiv \text{Im} \left[(\phi_1^* \mathbf{E}_1 + \mathbf{A}_1^* \times \mathbf{B}_1)/4\pi + \frac{1}{2} \int d^6z \delta^3(\mathbf{x} - \mathbf{r}) \left(\{S^*, f_0\} \mathbf{v} - e(\phi_1^* - \mathbf{v} \cdot \mathbf{A}_1^*/c) \frac{\partial f_0}{\partial \mathbf{p}} \right) S \right], \quad (16)$$

where we have used $\dot{S} = e(\phi_1 - \mathbf{v} \cdot \mathbf{A}_1/c)$, which follows from the full variation. We obtain the expressions for J and \mathbf{J} for the linearized Vlasov-Coulomb model by setting $\mathbf{A}_1 = 0$.

Just as for the cold-fluid model, one can take the eikonal limit, and then express only the *nonresonant* part of S in terms of the linear electromagnetic field. The local dispersion tensor is then Hermitian (in the conventional sense), while the *resonant* part of S appears explicitly. This formulation is useful in treating linear conversion between collective and ballistic modes [4–6].

In conclusion, we have derived wave-action conservation laws for several linear-wave systems, in terms of complex representations of the wave fields, which evolve by pseudo-Hermitian operators. The general procedure is as follows: From the variational functional $\mathcal{A}(\psi^*, \psi)$ in terms of the complexified field ψ , we perform the *restricted* variation $\delta\psi = i\alpha\psi$ and $\delta\psi^* = -i\alpha\psi^*$, and obtain the functional $\Delta\mathcal{A}(\psi^*, \psi; \alpha) \equiv \mathcal{A}(-i\alpha\psi^*, \psi) + \mathcal{A}(\psi^*, +i\alpha\psi)$. By construction, any terms in \mathcal{A} which do not involve space-time derivatives on ψ will not contribute to the wave-action conservation law. Then the action density $J(\mathbf{x}, t)$ is the functional derivative of $\Delta\mathcal{A}$ with respect to $\partial\alpha(\mathbf{x}, t)/\partial t$, while the flux \mathbf{J} is the functional derivative with respect to $\nabla\alpha$.

Because these wave-action conservation laws do not require eikonality, one can expect many applications, such as those listed in the opening paragraph.

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