# A new Lagrangian formulation for laser-plasma interactions

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A new Lagrangian structure for cold relativistic plasma electrodynamics is presented. This new formulation uses the fluid velocity **v** instead of the canonical-momentum Clebsch potential $\psi$ [X. L. Chen and R. N. Sudan, Phys. Fluids B **5**, 1336 (1993)]. As a simple application, it is used to derive (through the Noether method) new *exact* conservation laws associated with nonlinear laser wake-field equations in the multi-dimensional quasi-static approximation. © *1998 American Institute of Physics.* [S1070-664X(98)00104-9]

## I. INTRODUCTION

Small-scale laser wake-field acceleration (LWFA) schemes are presently being considered as substitutes for large-scale conventional particle accelerators (for recent reviews, see Refs. 1,2). The simplest LWFA scheme considers the self-consistent interaction between a high-power (high-frequency) laser pulse with a cold relativistic (underdense) electron plasma in the presence of a (neutralizing) nonuniform fixed-ion background (e.g., a plasma channel[3]). Since the laser high power induces large electron quiver velocities, the plasma electrons are treated relativistically and thermal effects are treated as higher-order effects and can be neglected.

In LWFA schemes, a plasma wakefield is generated by the displacement of electrons caused by the ponderomotive force associated with a high-frequency laser pulse propagating in an underdense plasma. Because of their large mass, the plasma ions are stationary and provide a neutralizing background which produces an electrostatic restoring force on the electrons. Since the laser field travels in the plasma at a speed close to the speed of light, the plasma wakefield can trap and accelerate electrons to high energies.

### A. Basic equations

The basic set of nonlinear equations describing the interaction of a laser pulse with a cold relativistic electron plasma includes the electron continuity equation

$$\frac{\partial n}{\partial t} = -\nabla\cdot n\mathbf{v}, \tag{1}$$

where $n$ is the electron plasma density and **v** is the electron fluid velocity; and the relativistic (kinetic) momentum equation

$$\frac{\partial \mathbf{p}}{\partial t} = -\mathbf{v}\cdot\nabla\mathbf{p} - e\left(\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}\right), \tag{2}$$

where the kinetic momentum is $\mathbf{p}=m\gamma\mathbf{v}$, with $\gamma\equiv(1-|\mathbf{v}/c|^2)^{-1/2}$, and $(-e,m)$ are the charge and mass of a single electron. In (2), **E** and **B** are the electric and magnetic fields associated with the electromagnetic potentials $(\phi,\mathbf{A})$: $\mathbf{E}=-\nabla\phi - c^{-1}\partial_t\mathbf{A}$ and $\mathbf{B}=\nabla\times\mathbf{A}$. For self-consistency, the Maxwell equations

$$0 = \nabla\cdot\mathbf{E} + 4\pi e(n-N), \tag{3}$$

$$\frac{\partial \mathbf{E}}{\partial t} = c\nabla\times\mathbf{B} + 4\pi e n\mathbf{v}, \tag{4}$$

are added to (1) and (2) to form the complete set (1)–(4) for the analysis of nonlinear laser-plasma interactions (within the simplest LWFA scheme). (The remaining two Maxwell equations follow directly from the definitions of **E** and **B** in terms of $\phi$ and **A**.) In (3), $N=N(\mathbf{r})$ represents the nonuniform fixed-ion background density; note that if (3) holds true initially then it holds true for all times, as dictated by (1) and (4).

### B. Lagrangian formulations

The set of nonlinear equations (1)–(4) has been extensively investigated in the past (see Ref. 1 for a historical survey). Because of their complexity, these equations are usually solved (analytically or numerically) by using a number of simplifying approximations. For example, the use of the eikonal approximation[4] for the laser field is justified by the fact that it has fast spatial and temporal scales compared to the plasma wake field and the plasma background. Following the standard approach, the implementation of such approximation schemes is typically done at the level of the equations themselves. Beside often involving tedious algebra, the standard approach offers no guiding principle on how truncation is to be carried out in order to preserve the conservation laws (e.g., energy and momentum) associated with (1)–(4). Yet these conservation laws can play an important role in checking numerical schemes used for solving (1)–(4) or approximate equations derived from them.

An alternative approach relies on the existence of a Lagrangian formulation for (1)–(4). In the Lagrangian approach, the implementation of an approximation scheme is done at the level of the Lagrangian itself (i.e., before using the variational principle), which greatly reduces the number of algebraic manipulations. Hence, one useful application of the Lagrangian approach is that it can be used to check the self-consistency of existing sets of approximate equations derived by the standard approach. In addition, any set of equations obtained from a truncated Lagrangian will automatically preserve the conservation laws of (1)–(4). Furthermore, the existence of a Lagrangian formulation for (1)–(4) allows the use of the powerful *averaged-Lagrangian* method[5] in studying the coupling between high-frequency laser field, the low-frequency plasma wake, and the plasma background. More generally, it also allows solutions of a given set of equations to be investigated without explicitly knowing them; for instance, conservation laws can used to discuss whether instabilities exist.[6]

The primary goal of this work is to present Lagrangian formulations for (1)–(4) as well as any approximate sets of equations derived from them. (Explicit applications of the Lagrangian formulation are beyond the scope of this work and are left for future work; see Sec. V for some possible applications.) Two Lagrangian formulations for (1)–(4) are presented in Sec. II. The first one, due to Chen and Sudan,[7] makes use of the Clebsch representation[8] for the canonical momentum $\mathbf{p}_c \equiv \nabla\psi$ in terms of the Clebsch potential $\psi$ [see (5) below] when the former satisfies $\nabla\times\mathbf{p}_c = 0$. In the Chen-Sudan formulation, the four variational fields $(n,\psi,\phi,\mathbf{A})$ are varied independently in the variational principle $\delta\int d^3x\,dt\,\mathscr{L}=0$, which then gives (1)–(4). Unfortunately, the presence of the unphysical potential $\psi$ prevents us from using the Noether method[9,10] for finding exact conservation laws for (1)–(4).

The second (new) Lagrangian formulation uses the electron-fluid velocity $\mathbf{v}$ as a variational field instead of the potential $\psi$. In the *constrained*-variational principle $\delta\int d^3x\,dt\,\mathscr{L}=0$, however, the variations $\delta n$ and $\delta\mathbf{v}$ are not independent[11] but are instead expressed in terms of the fluid displacement $\boldsymbol{\xi}$ [see (13.b) below]. In Sec. III, we show how our new formulation allows the use of the Noether method in deriving exact conservation laws associated with (1)–(4).

Although these two Lagrangian formulations are related [see (18)], the new formulation offers greater flexibility in deriving *reduced* Lagrangian formulations for approximate equations. We demonstrate this feature by deriving, in Sec. IV, a Lagrangian formulation for the nonlinear theory of intense laser-plasma interaction in the *multi-dimensional quasi-static approximation* (MDQS).[12] The Lagrangian formulation for a simpler set of (one-dimensional) quasi-static equations has already been constructed by Decker and Mori;[13] we recover their Lagrangian formulation in the appropriate limit. By applying the Noether method on the MDQS Lagrangian formulation, exact conservation laws corresponding to conservation of momentum, energy, and action are derived. In Sec. V, we summarize our work and discuss potential applications of this new Lagrangian formulation.

## II. LAGRANGIAN FORMULATIONS

The nonlinear equations (1)-(4) were recently shown to possess a Lagrangian formulation by Chen and Sudan;[7] their formulation relies on the use of Clebsch potentials.[8] After a brief presentation of the Chen-Sudan Lagrangian formulation, we present a new Lagrangian formulation based on the use of physical fields.

### A. Clebsch-potential variational principle

In the Clebsch-potential variational principle found by Chen and Sudan,[7] the variational fields are $(n,\psi,\phi,\mathbf{A})$, where $\psi$ is the electron fluid canonical-momentum potential:

$$\mathbf{p}_c \equiv m\gamma\mathbf{v} - e\mathbf{A}/c \equiv \nabla\psi. \tag{5}$$

Equations (1)–(4) are obtained from the variational principle $\delta\int d^3x\,dt\,\mathscr{L}_{CP}=0$, where the Clebsch-potential (CP) Lagrangian density is

$$\mathscr{L}_{CP} = \frac{1}{8\pi}(|\mathbf{E}|^2 - |\mathbf{B}|^2) - n\left[\frac{\partial\psi}{\partial t} + (\gamma-1)mc^2\right] + e(n-N)\phi, \tag{6}$$

where the relativistic factor $\gamma$ is

$$\gamma \equiv \sqrt{1 + |\nabla\psi + e\mathbf{A}/c|^2/(mc)^2}. \tag{7}$$

We note that the factor $(\gamma-1)$ appears in (6) instead of $\gamma$ so that, in the absence of the laser pulse ($\mathbf{A}=0$), the background plasma fields are $(n,\mathbf{v},\gamma,\phi)=(N,0,1,0)$ and the total energy functional vanishes [see (27.b) and Ref. 7 for details].

Chen and Sudan used the Lagrangian (6) to study the propagation of an intense laser pulse in a multi-dimensional background plasma. The existence of (6) allowed them to use the average Lagrangian method to study (1)–(4) in various limits. Their work revealed the importance of wave-dispersion effects on the laser-pulse propagation (which had been omitted in works prior to theirs).

The variational principle $\delta\int d^3x\,dt\,\mathscr{L}_{CP}=0$ under arbitrary variations $(\delta n,\delta\psi,\delta\phi,\delta\mathbf{A})$ yields the cold relativistic electron plasma electrodynamics equations as follows. Arbitrary variations in $\delta\psi$, $\delta\phi$, and $\delta\mathbf{A}$ yield

$$0 = \frac{\partial\mathscr{L}_{CP}}{\partial\psi} - \frac{\partial}{\partial t}\left(\frac{\partial\mathscr{L}_{CP}}{\partial(\partial_t\psi)}\right) - \nabla\cdot\left(\frac{\partial\mathscr{L}_{CP}}{\partial(\nabla\psi)}\right), \tag{8a}$$

$$0 = \frac{\partial\mathscr{L}_{CP}}{\partial\phi} + \nabla\cdot\left(\frac{\partial\mathscr{L}_{CP}}{\partial\mathbf{E}}\right), \tag{8b}$$

$$0 = \frac{\partial\mathscr{L}_{CP}}{\partial\mathbf{A}} + \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\partial\mathscr{L}_{CP}}{\partial\mathbf{E}}\right) + \nabla\times\left(\frac{\partial\mathscr{L}_{CP}}{\partial\mathbf{B}}\right), \tag{8c}$$

which, in turn, correspond to Eqs. (1), (3) and (4), respectively. For an arbitrary variation $\delta n$, on the other hand, we obtain

$$\frac{\partial\psi}{\partial t} = e\phi - (\gamma-1)mc^2. \tag{9}$$

To show that this equation is equivalent to Eq. (2), we transform (2) into an equation for the canonical momentum $\mathbf{p}_c \equiv \mathbf{p} - e\mathbf{A}/c$:

$$\frac{\partial\mathbf{p}_c}{\partial t} \equiv \nabla[e\phi - (\gamma-1)mc^2] + \mathbf{v}\times\nabla\times\mathbf{p}_c\,. \tag{10}$$

The evolution equation $\partial_t\boldsymbol{\Omega} = \nabla\times(\mathbf{v}\times\boldsymbol{\Omega})$ for $\boldsymbol{\Omega}\equiv\nabla\times\mathbf{p}_c$, obtained by taking the curl of (10), implies that if $\boldsymbol{\Omega}$ is initially zero, then it remains zero for all times. Hence, under this assumption (e.g., before the arrival of the laser pulse, we have $\mathbf{\Omega}=0$), by substituting $\mathbf{p}_c \equiv \nabla\psi$ into (10) we recover (9).

## B. New constrained variational principle

In the new constrained variational principle, the variations of the fluid fields $(n,\mathbf{v})$ are not arbitrary but are instead constrained by being expressed in terms of a *virtual* fluid displacement $\boldsymbol{\xi}$ [see (13.b)]. The advantage of this variational principle compared to the previous one is that $\mathbf{v}$, instead of the unphysical Clebsch potential $\psi$, appears explicitly. This situation allows a straightforward application of the Noether method[9] for the derivation of exact conservation laws.

Equations (1)–(4) are obtained from the variational principle $\delta\int d^3x\,dt\,\mathscr{L}_{CV}=0$, where the constrained-variation (CV) Lagrangian density is

$$\mathscr{L}_{CV} \equiv mnc^2(1-\gamma^{-1})+e(n-N)\phi-\frac{en\mathbf{v}}{c}\cdot\mathbf{A}+\frac{1}{8\pi}(|\mathbf{E}|^2-|\mathbf{B}|^2), \tag{11}$$

where $\gamma=(1-|\mathbf{v}|^2/c^2)^{-1/2}$. This Lagrangian is a simple generalization of the single-particle relativistic Lagrangian discussed, for example, in Ref. 14.

### 1. Gauge invariance

We point out that, for a Lagrangian density $\mathscr{L}(\phi,\mathbf{A},\ldots)$ which depends on the potentials $\phi$ and $\mathbf{A}$ explicitly, electromagnetic gauge invariance of the Lagrangian density requires that, under the gauge transformation:

$$\phi \rightarrow \phi - c^{-1}\partial\chi/\partial t \quad \text{and} \quad \mathbf{A}\rightarrow\mathbf{A}+\nabla\chi, \tag{12a}$$

where $\chi$ is an arbitrary scalar field, the Lagrangian density be independent of $\chi$, i.e.,

$$\frac{\partial\mathscr{L}}{\partial\chi}\equiv 0=\frac{\partial}{\partial t}\left(\frac{\partial\mathscr{L}}{\partial(\partial_t\chi)}\right)+\nabla\cdot\left(\frac{\partial\mathscr{L}}{\partial(\nabla\chi)}\right). \tag{12b}$$

This condition is satisfied when

$$\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\partial\mathscr{L}}{\partial\phi}\right)-\nabla\cdot\left(\frac{\partial\mathscr{L}}{\partial\mathbf{A}}\right)=0 \tag{12c}$$

is satisfied. We can readily check by substituting (11) that the gauge invariance condition (12.c) yields the particle number (or charge) conservation law (1).[10]

### 2. Constrained-variational principle

We now proceed with the variational derivation of (1)–(4) based on (11). An arbitrary variation of $\mathscr{L}(n,\mathbf{v},\phi,\mathbf{A},\mathbf{E},\mathbf{B})$ yields

$$\delta\mathscr{L}=\delta n\frac{\partial\mathscr{L}}{\partial n}+\delta\mathbf{v}\cdot\frac{\partial\mathscr{L}}{\partial\mathbf{v}}+\delta\phi\frac{\partial\mathscr{L}}{\partial\phi}+\cdots+\delta\mathbf{B}\cdot\frac{\partial\mathscr{L}}{\partial\mathbf{B}}, \tag{13a}$$

where the variations $\delta n$ and $\delta\mathbf{v}$ are not arbitrary but are instead expressed in terms of the *virtual* fluid displacement $\boldsymbol{\xi}$:[11]

$$\delta n\equiv-\nabla\cdot n\boldsymbol{\xi}, \quad \delta\mathbf{v}\equiv\frac{\partial\boldsymbol{\xi}}{\partial t}+\mathbf{v}\cdot\nabla\boldsymbol{\xi}-\boldsymbol{\xi}\cdot\nabla\mathbf{v}. \tag{13b}$$

Substituting these constraints into (13.a), with $\delta\mathbf{E}\equiv-\nabla\delta\phi-c^{-1}\partial_t\delta\mathbf{A}$ and $\delta\mathbf{B}\equiv\nabla\times\delta\mathbf{A}$, we obtain

$$\begin{aligned}\delta\mathscr{L}\equiv{}&\frac{\partial\mathscr{J}}{\partial t}+\nabla\cdot\mathbf{J}-\boldsymbol{\xi}\cdot\left[\frac{\partial}{\partial t}\left(\frac{\partial\mathscr{L}}{\partial\mathbf{v}}\right)+\nabla\cdot\left(\mathbf{v}\frac{\partial\mathscr{L}}{\partial\mathbf{v}}\right)+(\nabla\mathbf{v})\cdot\frac{\partial\mathscr{L}}{\partial\mathbf{v}}\right.\\&\left.-n\nabla\left(\frac{\partial\mathscr{L}}{\partial n}\right)\right]+\delta\phi\left[\frac{\partial\mathscr{L}}{\partial\phi}+\nabla\cdot\left(\frac{\partial\mathscr{L}}{\partial\mathbf{E}}\right)\right]\\&+\delta\mathbf{A}\cdot\left[\frac{\partial\mathscr{L}}{\partial\mathbf{A}}+\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\partial\mathscr{L}}{\partial\mathbf{E}}\right)+\nabla\times\left(\frac{\partial\mathscr{L}}{\partial\mathbf{B}}\right)\right],\end{aligned} \tag{14}$$

where the densities $\mathscr{J}$ and $\mathbf{J}$ are

$$\mathscr{J}\equiv\boldsymbol{\xi}\cdot\frac{\partial\mathscr{L}}{\partial\mathbf{v}}-\frac{\delta\mathbf{A}}{c}\cdot\frac{\partial\mathscr{L}}{\partial\mathbf{E}}, \tag{15a}$$

$$\mathbf{J}\equiv\mathbf{v}\boldsymbol{\xi}\cdot\frac{\partial\mathscr{L}}{\partial\mathbf{v}}-\boldsymbol{\xi}n\frac{\partial\mathscr{L}}{\partial n}-\delta\phi\frac{\partial\mathscr{L}}{\partial\mathbf{E}}+\delta\mathbf{A}\times\frac{\partial\mathscr{L}}{\partial\mathbf{B}}. \tag{15b}$$

Since $(\boldsymbol{\xi},\delta\phi,\delta\mathbf{A})$ are arbitrary, the variational principle $\delta\int d^3x\,dt\,\mathscr{L}_{CV}=0$ (where the spatial integration extends to infinity and the time integral is over a finite time interval) yields

$$0=\frac{\partial}{\partial t}\left(\frac{\partial\mathscr{L}_{CV}}{\partial\mathbf{v}}\right)+\nabla\cdot\left(\mathbf{v}\frac{\partial\mathscr{L}_{CV}}{\partial\mathbf{v}}\right)+(\nabla\mathbf{v})\cdot\frac{\partial\mathscr{L}_{CV}}{\partial\mathbf{v}}-n\nabla\left(\frac{\partial\mathscr{L}_{CV}}{\partial n}\right), \tag{16a}$$

$$0=\frac{\partial\mathscr{L}_{CV}}{\partial\phi}+\nabla\cdot\left(\frac{\partial\mathscr{L}_{CV}}{\partial\mathbf{E}}\right), \tag{16b}$$

$$0=\frac{\partial\mathscr{L}_{CV}}{\partial\mathbf{A}}+\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\partial\mathscr{L}_{CV}}{\partial\mathbf{E}}\right)+\nabla\times\left(\frac{\partial\mathscr{L}_{CV}}{\partial\mathbf{B}}\right), \tag{16c}$$

which correspond to (2), (3), and (4), respectively. [Recall that (1) is obtained from the gauge-invariance condition (12.c).] Here, it is understood that the variations $(\boldsymbol{\xi},\delta\phi,\delta\mathbf{A})$ vanish at infinity as well as at the time-integration boundaries, so that the surface terms coming from $\mathscr{J}$ and $\mathbf{J}$ vanish.

Since (16.a–c) must be true for all variations $(\boldsymbol{\xi},\delta\phi,\delta\mathbf{A})$, then (14) becomes

$$\delta\mathscr{L}_{CV}\equiv\frac{\partial\mathscr{J}}{\partial t}+\nabla\cdot\mathbf{J}. \tag{17}$$

This equation is the starting point of the application of the Noether method in deriving conservation laws (e.g., energy, linear and angular momenta, and wave action).

## C. Relation between the two Lagrangian formulations

Since the two Lagrangians (6) and (11) describe the same equations (when $\nabla\times\mathbf{p}_c=0$), there must exist a simple expression relating them. Indeed the relation between the Clebsch-potential (CP) Lagrangian density (6) and the constrained-variation (CV) Lagrangian density (11) is

$$\mathscr{L}_{CP} \equiv \mathscr{L}_{CV} + \psi\left(\frac{\partial n}{\partial t} + \nabla\cdot n\mathbf{v}\right) - \frac{\partial}{\partial t}(n\psi) - \nabla\cdot(\mathbf{v}n\psi), \tag{18}$$

where $\psi$ appears as a Lagrange multiplier and (5)–(7) must be used in verifying the identity. We note that, since the last two terms are exact derivatives, they do not modify the outcome of the variational principle $\delta\int d^3x\,dt\,\mathscr{L}=0$. We note that the (noncanonical) Hamiltonian structure for (1)–(4) was developed about ten years ago by Holm and Kupershmidt;[15] the Hamiltonian functional is common to both Lagrangians: $\mathscr{H} \equiv nmc^2(\gamma-1) + (|\mathbf{E}|^2 + |\mathbf{B}|^2)/8\pi$.

We note that finite-temperature effects, full ion dynamics, the presence of a beam-particle population, as well as any other dissipationless effect, can be easily included in these Lagrangian formulations. For simplicity, only the simplest LWFA model is considered in this paper.

## III. NOETHER SYMMETRIES AND CONSERVATION LAWS

Exact momentum and energy conservation laws can now be derived from (17) by the Noether method[9] as follows (additional conservation laws are discussed in Sec. III C). First, we introduce the infinitesimal generators for *uniform* temporal and spatial translations: $\sigma$ and $\boldsymbol{\eta}$, respectively. Next, we introduce

$$\boldsymbol{\xi} \equiv \boldsymbol{\eta} - \sigma\mathbf{v}, \tag{19a}$$

so that $\delta n$ and $\delta\mathbf{v}$ become

$$\delta n \equiv -\sigma\frac{\partial n}{\partial t} - \boldsymbol{\eta}\cdot\nabla n \quad \text{and} \quad \delta\mathbf{v} \equiv -\sigma\frac{\partial \mathbf{v}}{\partial t} - \boldsymbol{\eta}\cdot\nabla\mathbf{v}, \tag{19b}$$

while $\delta\phi$ and $\delta\mathbf{A}$ are given as

$$\delta\phi \equiv \boldsymbol{\eta}\cdot\mathbf{E} - \frac{\partial}{\partial t}\left(\sigma\phi - \frac{\boldsymbol{\eta}}{c}\cdot\mathbf{A}\right),$$

$$\delta\mathbf{A} \equiv \boldsymbol{\eta}\times\mathbf{B} + c\sigma\mathbf{E} + \nabla(c\sigma\phi - \boldsymbol{\eta}\cdot\mathbf{A}); \tag{19c}$$

we note that geometric interpretations for these expressions are available but are beyond the scope of this paper. To (19.a–c), we add

$$\delta\mathscr{L}_{CV} \equiv -\boldsymbol{\sigma}\left(\frac{\partial\mathscr{L}_{CV}}{\partial t} - \frac{\partial'\mathscr{L}_{CV}}{\partial t}\right) - \boldsymbol{\eta}\cdot(\nabla\mathscr{L}_{CV} - \nabla'\mathscr{L}_{CV}), \tag{20a}$$

where

$$\nabla'\mathscr{L} \equiv \nabla\mathscr{L} - \left(\nabla n\frac{\partial\mathscr{L}}{\partial n} + \nabla\mathbf{v}\cdot\frac{\partial\mathscr{L}}{\partial\mathbf{v}} + \cdots + \nabla\mathbf{B}\cdot\frac{\partial\mathscr{L}}{\partial\mathbf{B}}\right) \tag{20b}$$

and

$$\frac{\partial'\mathscr{L}}{\partial t} \equiv \frac{\partial\mathscr{L}}{\partial t} - \left(\frac{\partial n}{\partial t}\frac{\partial\mathscr{L}}{\partial n} + \frac{\partial\mathbf{v}}{\partial t}\cdot\frac{\partial\mathscr{L}}{\partial\mathbf{v}} + \cdots\frac{\partial\mathbf{B}}{\partial t}\cdot\frac{\partial\mathscr{L}}{\partial\mathbf{B}}\right) \tag{20c}$$

denote explicit spatial and temporal derivatives of the Lagrangian density $\mathscr{L}(\cdots;\mathbf{x},t)$. Substituting these expressions into (15.a and b), we find

$$\mathscr{J} = (\boldsymbol{\eta} - \sigma\mathbf{v})\cdot\frac{\partial\mathscr{L}}{\partial\mathbf{v}} - \left(\sigma\mathbf{E} + \frac{\boldsymbol{\eta}}{c}\times\mathbf{B}\right)\cdot\frac{\partial\mathscr{L}}{\partial\mathbf{E}} - \left(\sigma\phi - \frac{\boldsymbol{\eta}}{c}\cdot\mathbf{A}\right)\frac{\partial\mathscr{L}}{\partial\phi}, \tag{21a}$$

$$\mathbf{J} = \mathbf{v}(\boldsymbol{\eta} - \sigma\mathbf{v})\cdot\frac{\partial\mathscr{L}}{\partial\mathbf{v}} - (\boldsymbol{\eta} - \sigma\mathbf{v})n\frac{\partial\mathscr{L}}{\partial n} + (c\sigma\phi - \boldsymbol{\eta}\cdot\mathbf{A})\frac{\partial\mathscr{L}}{\partial\mathbf{A}} - \boldsymbol{\eta}\cdot\mathbf{E}\frac{\partial\mathscr{L}}{\partial\mathbf{E}} + (c\sigma\mathbf{E} + \boldsymbol{\eta}\times\mathbf{B})\times\frac{\partial\mathscr{L}}{\partial\mathbf{B}}, \tag{21b}$$

where exact derivatives and terms which vanish explicitly as a result of (16.a–c) are omitted.

### A. Momentum conservation law

Symmetry of a general Lagrangian density $\mathscr{L}(n,\ldots,\mathbf{B};\mathbf{x},t)$ with respect to an infinitesimal spatial translation $(\sigma=0,\boldsymbol{\eta}\neq 0)$ yields [from (17), (20.a), and (21.a and b)]

$$\frac{\partial\mathbf{G}}{\partial t} + \nabla\cdot\mathsf{T} = \nabla'\mathscr{L}, \tag{22a}$$

where

$$\mathbf{G} \equiv \frac{\partial\mathscr{L}}{\partial\mathbf{v}} + \frac{\mathbf{A}}{c}\frac{\partial\mathscr{L}}{\partial\phi} + \frac{\partial\mathscr{L}}{\partial\mathbf{E}}\times\frac{\mathbf{B}}{c}, \tag{22b}$$

is the momentum-density vector,

$$\mathsf{T} \equiv \mathbf{v}\frac{\partial\mathscr{L}}{\partial\mathbf{v}} + \mathbf{B}\frac{\partial\mathscr{L}}{\partial\mathbf{B}} - \frac{\partial\mathscr{L}}{\partial\mathbf{A}}\mathbf{A} - \frac{\partial\mathscr{L}}{\partial\mathbf{E}}\mathbf{E} + \mathbf{I}\left(\mathscr{L} - n\frac{\partial\mathscr{L}}{\partial n} - \mathbf{B}\cdot\frac{\partial\mathscr{L}}{\partial\mathbf{B}}\right), \tag{22c}$$

is the momentum-density-flux tensor. For the Lagrangian density (11), we find

$$\mathbf{G} = mn\gamma\mathbf{v} + \frac{\mathbf{E}\times\mathbf{B}}{4\pi c} - \frac{eN}{c}\mathbf{A}, \tag{23a}$$

$$\mathsf{T} = mn\gamma\mathbf{v}\mathbf{v} - \frac{1}{4\pi}\left[\mathbf{E}\mathbf{E} + \mathbf{B}\mathbf{B} - \frac{1}{2}(|\mathbf{E}|^2 + |\mathbf{B}|^2)\mathbf{I}\right] - eN\phi\mathbf{I}, \tag{23b}$$

and

$$\nabla'\mathscr{L}_{CV} = -e\phi\nabla N, \tag{23c}$$

so that (22.a) becomes

$$\frac{\partial\mathbf{G}}{\partial t} + \nabla\cdot\mathsf{T} = -e\phi\nabla N. \tag{24}$$

If the ion background density $N$ is invariant along a particular direction, then the component of $\mathbf{G}$ in that direction is conserved (this is the Noether theorem). Moreover, according to (24), a finite background-density gradient $\nabla N$, when coupled to the plasma wake field $\phi$ (due to an electron density perturbation), causes the total momentum density $\mathbf{G}$ in the direction of the density gradient to change. This in turn indicates how transfer of momentum between the laser pulse and the plasma wake (or the electrons) is affected by the plasma background (as was mentioned in Ref. 16).

We note that the Noether equation (24), with $\mathbf{G}$ and $\mathsf{T}$ given by (23.a and b), is gauge-invariant (although not manifestly so). A different form for (24) which is manifestly gauge-invariant can be obtained by defining

$$\mathbf{G} \equiv \mathbf{G}' - \frac{eN}{c}\mathbf{A} \quad \text{and} \quad \mathsf{T} \equiv \mathsf{T}' - eN\phi\mathbf{I}, \tag{25a}$$

so that (24) becomes

$$\frac{\partial \mathbf{G}'}{\partial t} + \nabla \cdot \mathsf{T}' = -eN\mathbf{E}. \tag{25b}$$

This equation implies that if $\mathbf{E}$ vanishes everywhere in a particular direction, then the component of $\mathbf{G}'$ in that direction is also conserved. Equation (25.b) could also be used to study the generation of quasi-static magnetic fields in laser-plasma interactions.

### B. Energy conservation law

Next, symmetry of the Lagrangian density $\mathscr{L}(n,\ldots,\mathbf{B};\mathbf{x},t)$ with respect to an infinitesimal time translation $(\sigma \neq 0, \boldsymbol{\eta}=0)$ yields [from (17), (20.a), and (21.a and b)]

$$\frac{\partial E}{\partial t} + \nabla \cdot \mathbf{S} = -\frac{\partial' \mathscr{L}}{\partial t}, \tag{26a}$$

where

$$E \equiv \mathbf{v} \cdot \frac{\partial \mathscr{L}}{\partial \mathbf{v}} + \phi \frac{\partial \mathscr{L}}{\partial \phi} + \mathbf{E} \cdot \frac{\partial \mathscr{L}}{\partial \mathbf{E}} - \mathscr{L} \tag{26b}$$

is the energy density,

$$\mathbf{S} \equiv c\frac{\partial \mathscr{L}}{\partial \mathbf{B}} \times \mathbf{E} - c\phi \frac{\partial \mathscr{L}}{\partial \mathbf{A}} + \mathbf{v}\left(\mathbf{v} \cdot \frac{\partial \mathscr{L}}{\partial \mathbf{v}} - n\frac{\partial \mathscr{L}}{\partial n}\right), \tag{26c}$$

is the energy-density flux. By substituting (11) into (26.b and c) (while using the fact that the fixed-ion background density is time independent, i.e., $\partial_t' \mathscr{L}_{CV} = -e\phi \partial_t N \equiv 0$), one obtains the energy conservation law:

$$\frac{\partial E}{\partial t} + \nabla \cdot \mathbf{S} = 0, \tag{27a}$$

where

$$E = mc^2 n(\gamma - 1) + \frac{1}{8\pi}(|\mathbf{E}|^2 + |\mathbf{B}|^2), \tag{27b}$$

$$\mathbf{S} = mc^2 n(\gamma - 1)\mathbf{v} + \frac{c}{4\pi}\mathbf{E} \times \mathbf{B}. \tag{27c}$$

### C. Other conservation laws

Other conservation laws associated with the continuous symmetries of a Lagrangian density include: (a) the conservation of angular momentum associated with rotational symmetry; (b) the conservation of helicity (defined as $h \equiv \mathbf{p}_c \cdot \nabla \times \mathbf{p}_c$) associated with relabeling symmetry (for fluid systems);[17] and (c) the conservation of wave action associated with phase-shift symmetry.[18]

The latter symmetry is associated with a Lagrangian density $\mathscr{L}$ which depends on a *complex*-valued multi-component field $\Psi_i$ $(i=1,\ldots,N)$ and its derivatives (expressed as $\partial \Psi_i / \partial x^\mu$, where $x^\mu$ denotes either time $t$ or spatial coordinates $\mathbf{x}$). The invariance of the real-valued Lagrangian $\mathscr{L}$ under the phase-shift transformation

$$\Psi_i \rightarrow \Psi_i e^{i\epsilon}, \tag{28a}$$

where $\epsilon \ll 1$ is a real infinitesimal phase shift, leads to the conservation law:

$$0 = \frac{\partial}{\partial x^\mu} \mathrm{Im}\left[\Psi_i(\mathbf{x},t)\frac{\partial \mathscr{L}}{\partial(\partial \Psi_i / \partial x^\mu)}(\mathbf{x},t)\right], \tag{28b}$$

where sum over repeated indices is implied. In previous work,[18–20] we have found that when the eikonal representation for $\Psi_i$ is used, (28.b) corresponds to the conservation of wave action. Conservation of wave action is considered explicitly in the next section (see Sec. IV D 2).

## IV. NONLINEAR EQUATIONS IN THE QUASI-STATIC APPROXIMATION

In this section, we consider one application of the new Lagrangian formulation (11). We derive a Lagrangian formulation for the nonlinear laser-plasma wake equations in the multi-dimensional quasi-static (MDQS) approximation[12,21] [these equations were initially derived by applying the MDQS approximation on the nonlinear equations (1)–(4)]. The existence of a Lagrangian formulation for the MDQS equations points to their self-consistency and ensures that they preserve all of the important conservation laws which the original equations possess.

The Lagrangian formulation for the simplest set of (one-dimensional) QS nonlinear equations[22] has already been constructed *a fortiori* by Decker and Mori.[13] It is recovered here from our MDQS Lagrangian density in the one-dimensional QS (1DQS) limit [see (34.c)]. Decker and Mori then used the Lagrangian (34.c) to investigate the propagation of a large-amplitude laser pulse in a one-dimensional underdense unmagnetized plasma. The new MDQS Lagrangian (33) could be used to extend this work to two and three dimensions (such an application is, however, outside the scope of this work).

### A. MDQS Lagrangian density

In the MDQS model,[12] the evolution equations for the fields $(n,\mathbf{v},\phi,\mathbf{A})$ are written in terms of the *light*-frame coordinates $(\zeta,\mathbf{x}_\perp,\tau)$, where $\zeta \equiv z - ct$, $\mathbf{x}_\perp \equiv (x,y)$, and $\tau \equiv t$. In these coordinates, $\nabla \equiv \nabla_\perp + \hat{z}\partial/\partial\zeta$, $\nabla^2 \equiv \nabla_\perp^2 + \partial^2/\partial\zeta^2$, and $\partial/\partial t = \partial/\partial\tau - c\,\partial/\partial\zeta$. Next, to facilitate comparison with previous QS models, we introduce the following normalization: $\rho \equiv n/N$, $\mathbf{u} \equiv \gamma\mathbf{v}/c$, $\varphi \equiv e\phi/mc^2$, and $\mathbf{a} \equiv e\mathbf{A}/mc^2$; as in Ref. 12, we also consider a uniform background density $N$.

We now break up each field $(\rho,\mathbf{u},\varphi,\mathbf{a})$ in terms of its low-frequency wakefield component (denoted by an overbar) and its high-frequency laser-field component (denoted by a tilde). Hence, we use

$$
\begin{aligned}
\rho &= \overline{\rho}, \\
\mathbf{u} &= \overline{\mathbf{u}} + \widetilde{\mathbf{a}}, \\
\varphi &= \overline{\varphi}, \\
\mathbf{a} &= \overline{\mathbf{a}} + \widetilde{\mathbf{a}},
\end{aligned} \tag{29a}
$$

where the term $\widetilde{\mathbf{a}}$ in $\mathbf{u}$ represents the relativistic electron quiver velocity. We assume that both $\overline{\mathbf{a}}$ and $\widetilde{\mathbf{a}}$ are divergenceless and that $\hat{z}\cdot\widetilde{\mathbf{a}}=0$. We further assume that $\widetilde{\mathbf{a}}$ has circular polarization, i.e., $\widetilde{\mathbf{a}} \equiv (\hat{\mathbf{a}}e^{i\theta}+\hat{\mathbf{a}}^{*}e^{-i\theta})/2$, where $\theta$ is the *fast* laser eikonal phase and the eikonal amplitudes $\hat{\mathbf{a}}$ and $\hat{\mathbf{a}}^{*}$ satisfy $\hat{\mathbf{a}}\cdot\hat{\mathbf{a}}=0=\hat{\mathbf{a}}^{*}\cdot\hat{\mathbf{a}}^{*}$ and $\hat{\mathbf{a}}\cdot\hat{\mathbf{a}}^{*}\equiv|\hat{\mathbf{a}}|^{2}$, so that $|\widetilde{\mathbf{a}}|^{2}$ is independent of the fast laser time scale (i.e., $|\widetilde{\mathbf{a}}|^{2}$ is $\theta$-independent). Hence, the relativistic factor has only a low-frequency component:

$$
\gamma = (1+|\overline{\mathbf{u}}|^{2}+|\widetilde{\mathbf{a}}|^{2})^{1/2} \equiv \overline{\gamma}. \tag{29b}
$$

In addition, in the MDQS approximation,[12] the wakefields $(\overline{\rho},\overline{\mathbf{u}},\overline{\phi},\overline{\mathbf{a}})$ are functions of $(\zeta,\mathbf{x}_{\perp})$ only (i.e., $\partial_{\tau}$ vanishes whenever it acts on wakefields). Thus, when these expressions are substituted into the Lagrangian density (11), we obtain

$$
\begin{aligned}
\mathscr{L} = {} & k_p^2[\overline{\rho}(1+\overline{\varphi}-\overline{\gamma}^{-1}(1+\overline{\mathbf{u}}\cdot\overline{\mathbf{a}}+|\widetilde{\mathbf{a}}|^{2}))-\overline{\varphi}] \\
& + \frac{1}{2}\left( \left| \nabla\overline{\varphi} - \frac{\partial\overline{\mathbf{a}}}{\partial\zeta} \right|^{2} - |\nabla\times\overline{\mathbf{a}}|^{2} \right) \\
& + \frac{1}{2}\left( \frac{1}{c^{2}} \left| \frac{\partial\widetilde{\mathbf{a}}}{\partial\tau} - c\frac{\partial\widetilde{\mathbf{a}}}{\partial\zeta} \right|^{2} - |\nabla\times\widetilde{\mathbf{a}}|^{2} \right),
\end{aligned} \tag{30}
$$

where $k_p^2 \equiv 4\pi Ne^2/mc^2$.

Next, the wakefield equations [obtained from (30)]

$$
\frac{\partial}{\partial\zeta}\left[ \overline{\rho}\left(1-\frac{\overline{u}_{\parallel}}{\overline{\gamma}}\right)\right] = \nabla_{\perp}\cdot\left(\overline{\rho}\frac{\overline{\mathbf{u}}_{\perp}}{\overline{\gamma}}\right), \tag{31a}
$$

$$
\frac{\partial}{\partial\zeta}(\overline{\mathbf{u}}-\overline{\mathbf{a}}) = \nabla(\overline{\gamma}-\overline{\varphi}), \tag{31b}
$$

$$
\left(\nabla_{\perp}^{2}+\frac{\partial^{2}}{\partial\zeta^{2}}\right)\overline{\varphi} = k_p^2(\overline{\rho}-1), \tag{31c}
$$

$$
\nabla_{\perp}^{2}\overline{\mathbf{a}} = \frac{k_p^2\overline{\rho}}{\overline{\gamma}}\overline{\mathbf{u}} - \frac{\partial}{\partial\zeta}\nabla\overline{\varphi}, \tag{31d}
$$

can be used to solve for $(\overline{\rho},\overline{\mathbf{u}},\overline{\mathbf{a}})$:[12]

$$
\overline{\rho} = \frac{1+k_p^{-2}\nabla_{\perp}^{2}\overline{\chi}}{1-\overline{u}_{\parallel}/\overline{\gamma}} = \overline{\gamma}\frac{1+k_p^{-2}\nabla_{\perp}^{2}\overline{\chi}}{1+\overline{\chi}}, \tag{32a}
$$

$$
\begin{aligned}
\overline{u}_{\parallel} &= -\frac{1}{2}\left[1+\overline{\chi}-\frac{(1+|\overline{\mathbf{u}}_{\perp}|^{2}+|\widetilde{\mathbf{a}}|^{2})}{1+\overline{\chi}}\right], \\
\overline{\mathbf{u}}_{\perp} &= \frac{\overline{\gamma}}{\overline{\rho}k_p^2}\nabla_{\perp}\frac{\partial\overline{\chi}}{\partial\zeta},
\end{aligned} \tag{32b}
$$

$$
\nabla^{2}\overline{a}_{\parallel} = \nabla_{\perp}^{2}\overline{u}_{\parallel} - \nabla_{\perp}\cdot\frac{\partial\overline{\mathbf{u}}_{\perp}}{\partial\zeta}, \quad \nabla_{\perp}\cdot\overline{\mathbf{a}}_{\perp} = -\frac{\partial\overline{a}_{\parallel}}{\partial\zeta}, \tag{32c}
$$

where

$$
\overline{\chi} \equiv \overline{\varphi} - \overline{a}_{\parallel} \equiv \overline{\gamma} - 1 - \overline{u}_{\parallel}. \tag{32d}
$$

From the condition $\nabla\cdot\overline{\mathbf{a}}=0$, we may write $\overline{a}_{\parallel}\equiv\nabla_{\perp}^{2}\overline{f}$ and $\overline{\mathbf{a}}_{\perp}\equiv-\nabla_{\perp}\partial_{\zeta}\overline{f}$ (where $\overline{f}$ is a scalar field), so that $\nabla\times\overline{\mathbf{a}}\equiv(\nabla_{\perp}\overline{a}_{\parallel}-\partial_{\zeta}\overline{\mathbf{a}}_{\perp})\times\hat{z}$. Likewise, from the conditions $\nabla\cdot\widetilde{\mathbf{a}}=0$ and $\hat{z}\cdot\widetilde{\mathbf{a}}=0$, we find $\nabla\times\widetilde{\mathbf{a}}\equiv\hat{z}\widetilde{b}_{\parallel}+\hat{z}\times\partial_{\zeta}\widetilde{\mathbf{a}}$, where $\widetilde{b}_{\parallel}\equiv\hat{z}\cdot\nabla_{\perp}\times\widetilde{\mathbf{a}}$.

Substituting these expressions into (30), we obtain the MDQS Lagrangian density

$$
\begin{aligned}
\mathscr{L}_{\mathrm{MDQS}} \equiv {} & \frac{1}{2}|\nabla\overline{\chi}|^{2} - k_p^2\overline{\chi} + \overline{\mathbf{u}}_{\perp}(\overline{\chi})\cdot\nabla_{\perp}\frac{\partial\overline{\chi}}{\partial\zeta} \\
& - (k_p^2+\nabla_{\perp}^{2}\overline{\chi})\overline{u}_{\parallel}(\overline{\chi},\widetilde{\mathbf{a}}) \\
& + \frac{1}{2}\left(\frac{1}{c^{2}}\left|\frac{\partial\widetilde{\mathbf{a}}}{\partial\tau}\right|^{2} - \frac{2}{c}\frac{\partial\widetilde{\mathbf{a}}}{\partial\tau}\cdot\frac{\partial\widetilde{\mathbf{a}}}{\partial\zeta} - (\widetilde{b}_{\parallel})^{2}\right).
\end{aligned} \tag{33}
$$

We note from (32.b) that $\overline{u}_{\parallel}$ depends explicitly on the wakefield $\overline{\chi}$ (and its derivatives) and the laser field $\widetilde{\mathbf{a}}$ (through $|\widetilde{\mathbf{a}}|^{2}$), while $\overline{\mathbf{u}}_{\perp}$ depends only on $\overline{\chi}$ (and its derivatives).

### B. One-dimensional QS limit

In the one-dimensional QS (1DQS) limit,[22] the spatial dependence of the fields involves only $\zeta$, i.e., inserting $\nabla_{\perp}=0$ in (32.a–c), we obtain

$$
\widetilde{b}_{\parallel}\equiv 0, \quad \overline{\mathbf{a}}\equiv 0, \text{ and } \overline{\mathbf{u}}_{\perp}\equiv 0. \tag{34a}
$$

From (32.a and b), we obtain

$$
\overline{\rho}\equiv(1-\overline{\beta})^{-1}, \quad \overline{u}_{\parallel}\equiv\overline{\gamma\beta}, \text{ and } \overline{\gamma}\equiv(1+\overline{\varphi})\overline{\rho}. \tag{34b}
$$

Substituting these expressions into the MDQS Lagrangian density (33), we obtain the 1DQS Lagrangian density

$$
\begin{aligned}
\mathscr{L}_{\mathrm{1DQS}} \equiv {} & \frac{1}{2}\left[\left(\frac{\partial\overline{\varphi}}{\partial\zeta}\right)^{2} + k_p^2\left(1-\overline{\varphi}-\frac{1+|\widetilde{\mathbf{a}}|^{2}}{1+\overline{\varphi}}\right) + \frac{1}{c^{2}}\left|\frac{\partial\widetilde{\mathbf{a}}}{\partial\tau}\right|^{2} \right. \\
& \left. - \frac{2}{c}\frac{\partial\widetilde{\mathbf{a}}}{\partial\tau}\cdot\frac{\partial\widetilde{\mathbf{a}}}{\partial\zeta}\right],
\end{aligned} \tag{34c}
$$

from which the nonlinear coupled equations for $\overline{\phi}$ and $\widetilde{\mathbf{a}}$ are derived. This Lagrangian was previously constructed by Decker and Mori[13] and used in their study of the propagation of an intense laser pulse in an underdense unmagnetized plasma. It is derived here from first principles by introducing the 1DQS approximation (34.a and b) into the Lagrangian density (33).

### C. MDQS variational principle

We now proceed to show that the MDQS nonlinear equations for $\overline{\chi}$ and $\widetilde{\mathbf{a}}$ [see (38.a and b) below] are obtained from the variational principle $\delta\int d^{2}x_{\perp}\,d\zeta\,d\tau\,\mathscr{L}_{\mathrm{MDQS}}=0$.

An arbitrary variation of (33) yields

$$\delta\mathscr{L}_{\mathrm{MDQS}}=\frac{\partial\mathscr{J}}{\partial\tau}+\frac{\partial J_\parallel}{\partial\zeta}+\nabla_\perp\cdot\mathbf{J}_\perp - \delta\overline{\chi}\left[\frac{\partial^2\overline{\chi}}{\partial\zeta^2}-\left(\frac{k_p^2\overline{\rho}}{\overline{\gamma}}-\nabla_\perp^2\right)\overline{u}_\parallel-\nabla_\perp\cdot\frac{\partial\overline{\mathbf{u}}_\perp}{\partial\zeta}\right] - \delta\widetilde{\mathbf{a}}\cdot\left[\frac{k_p^2\overline{\rho}}{\overline{\gamma}}\widetilde{\mathbf{a}}+\left(\frac{1}{c^2}\frac{\partial^2}{\partial\tau^2}-\frac{2}{c}\frac{\partial^2}{\partial\tau\partial\zeta}-\nabla_\perp^2\right)\widetilde{\mathbf{a}}\right], \tag{35}$$

where the following identity [obtained from (32.b)]

$$\delta\overline{\mathbf{u}}_\perp\cdot\nabla_\perp\frac{\partial\overline{\chi}}{\partial\zeta}-(k_p^2+\nabla_\perp^2\overline{\chi})\,\delta\overline{u}_\parallel \equiv\left[(k_p^2+\nabla_\perp^2\overline{\chi})+\frac{k_p^2\overline{\rho}}{\overline{\gamma}}\overline{u}_\parallel\right]\delta\overline{\chi}-\left(\frac{k_p^2\overline{\rho}}{\overline{\gamma}}\widetilde{\mathbf{a}}\right)\cdot\delta\widetilde{\mathbf{a}} \tag{36}$$

is used, and the densities $\mathscr{J}$, $J_\parallel$, and $\mathbf{J}_\perp$ are defined as

$$\mathscr{J}\equiv\frac{\delta\widetilde{\mathbf{a}}}{c^2}\cdot\left(\frac{\partial\widetilde{\mathbf{a}}}{\partial\tau}-c\frac{\partial\widetilde{\mathbf{a}}}{\partial\zeta}\right), \tag{37a}$$

$$J_\parallel\equiv\delta\overline{\chi}\left(\frac{\partial\overline{\chi}}{\partial\zeta}-\nabla_\perp\cdot\overline{\mathbf{u}}_\perp\right)-\frac{\delta\widetilde{\mathbf{a}}}{c}\cdot\frac{\partial\widetilde{\mathbf{a}}}{\partial\tau}, \tag{37b}$$

$$\mathbf{J}_\perp\equiv\delta\overline{\chi}\nabla_\perp\overline{\gamma}+\overline{\mathbf{u}}_\perp\frac{\partial\delta\overline{\chi}}{\partial\zeta}-\overline{u}_\parallel\nabla_\perp\delta\overline{\chi}-\delta\widetilde{\mathbf{a}}\times\hat{z}\widetilde{b}_\parallel. \tag{37c}$$

From (35), the variational principle $\delta\int d^2x_\perp\,d\zeta\,d\tau\,\mathscr{L}_{\mathrm{MDQS}}=0$ yields, for arbitrary variations $\delta\overline{\chi}$ and $\delta\widetilde{\mathbf{a}}$ (which vanish at the integration boundaries so that the surface terms coming from $\mathscr{J}$, $J_\parallel$, and $\mathbf{J}_\perp$ vanish), the nonlinear MDQS laser-wakefield equations:[12]

$$\left(\nabla_\perp^2+\frac{2}{c}\frac{\partial^2}{\partial\tau\partial\zeta}-\frac{1}{c^2}\frac{\partial^2}{\partial\tau^2}\right)\widetilde{\mathbf{a}}=\frac{\overline{\rho}k_p^2}{\overline{\gamma}}\widetilde{\mathbf{a}}, \tag{38a}$$

$$\frac{\partial^2\overline{\chi}}{\partial\zeta^2}=\left(\frac{\overline{\rho}k_p^2}{\overline{\gamma}}-\nabla_\perp^2\right)\overline{u}_\parallel+\frac{\partial}{\partial\zeta}(\nabla_\perp\cdot\overline{\mathbf{u}}_\perp), \tag{38b}$$

Since these equations are valid for arbitrary variations $\delta\overline{\chi}$ and $\delta\widetilde{\mathbf{a}}$, (35) becomes

$$\delta\mathscr{L}_{QS}=\frac{\partial\mathscr{J}}{\partial\tau}+\frac{\partial J_\parallel}{\partial\zeta}+\nabla_\perp\cdot\mathbf{J}_\perp. \tag{39}$$

This equation then becomes the starting point for the application of the Noether method in deriving *exact* conservation laws (as demonstrated in the previous section for the full equations).

## D. Exact MDQS conservation laws

The analytical and numerical solutions of the nonlinear equations (38.a and b) represent a difficult task. The existence of conservation laws associated with these equations is often quite crucial in obtaining reliable results. Because (38.a and b) possess a Lagrangian formulation, *exact* conservation laws (within the MDQS approximation) can be derived by applying the Noether method on the Lagrangian density (33).

### 1. Energy and parallel momentum conservation laws

We first derive the energy and parallel momentum conservation laws. Substituting the light-frame coordinates $(\zeta,\mathbf{x}_\perp,\tau)$ into the variations (19.c), we find

$$\delta\overline{\chi}=(c\sigma-\eta_\parallel)\frac{\partial\overline{\chi}}{\partial\zeta}, \tag{40a}$$

$$\delta\widetilde{\mathbf{a}}=-\sigma\frac{\partial\widetilde{\mathbf{a}}}{\partial\tau}+(c\sigma-\eta_\parallel)\frac{\partial\widetilde{\mathbf{a}}}{\partial\zeta}, \tag{40b}$$

where $\sigma$ and $\eta_\parallel$ are generators for infinitesimal temporal and spatial (along the $\zeta$-axis) translations. Moreover, we have

$$\delta\mathscr{L}_{\mathrm{MDQS}}=-\sigma\frac{\partial\mathscr{L}_{\mathrm{MDQS}}}{\partial\tau}+(c\sigma-\eta_\parallel)\frac{\partial\mathscr{L}_{\mathrm{MDQS}}}{\partial\zeta}, \tag{40c}$$

where the $\tau$ partial derivative acts only on the laser-field terms in (33). After some algebra, recognizing that the transformations $J_\parallel\to J_\parallel-\nabla_\perp\cdot\mathbf{\Gamma}_\perp$ and $\mathbf{J}_\perp\to\mathbf{J}_\perp+\partial_\zeta\mathbf{\Gamma}_\perp$ leave the right side of (39) invariant, and using the identity (36), with $\delta(\cdots)$ replaced by $\partial_\zeta(\cdots)$, we obtain the *exact* energy ($\sigma\neq0$) conservation law:

$$\frac{\partial\mathscr{E}}{\partial\tau}=\frac{\partial}{\partial\zeta}\left[\frac{1}{2c}\left|\frac{\partial\widetilde{\mathbf{a}}}{\partial\tau}\right|^2+\frac{c}{2}(\widetilde{b}_\parallel)^2\right]+\nabla_\perp\cdot\left[\widetilde{b}_\parallel\left(\frac{\partial\widetilde{\mathbf{a}}}{\partial\tau}-c\frac{\partial\widetilde{\mathbf{a}}}{\partial\zeta}\right)\times\hat{z}\right] +\frac{ck_p^2\overline{\rho}}{2\overline{\gamma}}\frac{\partial|\widetilde{\mathbf{a}}|^2}{\partial\zeta}, \tag{41a}$$

where $\mathscr{E}$ is the MDQS energy density:

$$\mathscr{E}\equiv\frac{1}{2}\left(\left|\frac{1}{c}\frac{\partial\widetilde{\mathbf{a}}}{\partial\tau}-\frac{\partial\widetilde{\mathbf{a}}}{\partial\zeta}\right|^2+|\nabla\times\widetilde{\mathbf{a}}|^2\right), \tag{41b}$$

which is the dimensionless form of the laser field energy $(|\mathbf{E}|^2+|\mathbf{B}|^2)/8\pi$ [see (27.b)].

The *exact* parallel momentum ($\eta_\parallel\neq0$) conservation law, on the other hand, is

$$\frac{\partial\mathscr{P}_\parallel}{\partial\tau}=-\frac{\partial}{\partial\zeta}\left[\frac{1}{2c^2}\left|\frac{\partial\widetilde{\mathbf{a}}}{\partial\tau}\right|^2-\frac{1}{2}(\widetilde{b}_\parallel)^2\right]+\nabla_\perp\cdot\left(\widetilde{b}_\parallel\hat{z}\times\frac{\partial\widetilde{\mathbf{a}}}{\partial\zeta}\right) +\frac{k_p^2\overline{\rho}}{2\overline{\gamma}}\frac{\partial|\widetilde{\mathbf{a}}|^2}{\partial\zeta}, \tag{42a}$$

where $\mathscr{P}_\parallel$ is the MDQS parallel momentum density:

$$\mathscr{P}_\parallel\equiv-\frac{1}{c}\frac{\partial\widetilde{\mathbf{a}}}{\partial\zeta}\cdot\left(\frac{1}{c}\frac{\partial\widetilde{\mathbf{a}}}{\partial\tau}-\frac{\partial\widetilde{\mathbf{a}}}{\partial\zeta}\right), \tag{42b}$$

which is the dimensionless form of the laser parallel momentum $\hat{z}\cdot(\mathbf{E}\times\mathbf{B})/4\pi c$ [see (23.a)].

Looking at (41.a) and (42.a), we see how ponderomotive effects (represented by $\partial_\zeta|\widetilde{\mathbf{a}}|^2$) induce transfer of energy and parallel momentum between the laser pulse and the plasma wake. We note, however, that the quantity $\mathscr{E}-c\mathscr{P}_\parallel$ is conserved:

$$\frac{\partial}{\partial\tau}(\mathscr{E}-c\mathscr{P}_{\parallel})=\frac{\partial}{\partial\zeta}\left(\frac{1}{c^2}\left|\frac{\partial\tilde{\mathbf{a}}}{\partial\tau}\right|^2\right)+\nabla_\perp\cdot\left[\tilde{b}_{\parallel}\left(\frac{\partial\tilde{\mathbf{a}}}{\partial\tau}-2c\,\frac{\partial\tilde{\mathbf{a}}}{\partial\zeta}\right)\times\hat{z}\right]. \tag{42c}$$

in analogy to the single-particle case.[21]

### 2. Wave action conservation law

Next, we derive a third conservation law from (39) by allowing the complexification of the laser field $\tilde{\mathbf{a}}$,[18] while $\overline{\chi}$ remains real (see discussion in Sec. III C). Now consider the variations of $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{a}}^*$ under the infinitesimal phase changes $\tilde{\mathbf{a}}\rightarrow\tilde{\mathbf{a}}e^{i\epsilon}$ and $\tilde{\mathbf{a}}^*\rightarrow\tilde{\mathbf{a}}^*e^{-i\epsilon}$, where $\epsilon$ is real and $\epsilon\ll 1$. Thus, $\delta\tilde{\mathbf{a}}=i\epsilon\tilde{\mathbf{a}}$, $\delta\tilde{\mathbf{a}}^*=-i\epsilon\tilde{\mathbf{a}}^*$, and, since $\mathscr{L}_{\mathrm{MDQS}}$ and $\overline{\chi}$ are both real, $\delta\mathscr{L}_{\mathrm{MDQS}}=0=\delta\overline{\chi}$. Substituting these expressions in (39) yields the following *exact* wave-action conservation law for the laser field:

$$\frac{\partial\mathscr{A}}{\partial\tau}=\frac{\partial}{\partial\zeta}\mathrm{Im}\left(\frac{\tilde{\mathbf{a}}}{c}\cdot\frac{\partial\tilde{\mathbf{a}}^*}{\partial\tau}\right)+\nabla_\perp\cdot\mathrm{Im}(\tilde{b}_{\parallel}^*\tilde{\mathbf{a}}\times\hat{z}), \tag{43}$$

where the wave-action density $\mathscr{A}$ is

$$\mathscr{A}\equiv\mathrm{Im}\left[\frac{\tilde{\mathbf{a}}}{c}\cdot\left(\frac{\partial\tilde{\mathbf{a}}^*}{\partial\tau}-c\,\frac{\partial\tilde{\mathbf{a}}^*}{\partial\zeta}\right)\right], \tag{44}$$

which is the dimensionless form of $\mathrm{Im}(\mathbf{A}^*\cdot\mathbf{E}/4\pi c)$. Using the eikonal representation for $\mathbf{A}$ (with $\mathbf{E}=i\omega\mathbf{A}/c$), this last expression becomes $\omega^{-1}|\mathbf{E}|^2/4\pi$, which is indeed the action of an electromagnetic wave.

## V. CONCLUSIONS

We now summarize our results and discuss extensions of this work. Two Lagrangian formulations were presented for the cold relativistic electron plasma electrodynamic equations (1)–(4). The first formulation,[7] based on the Lagrangian density (6), relied on the use of the Clebsch representation for the electron-fluid canonical momentum. The second formulation, based on the Lagrangian density (11), is the primary new result of this work. Its associated variational principle makes use of constrained variations for the electron-fluid density and velocity. The Noether method was then applied on the new Lagrangian density and exact conservation laws for energy and momentum were derived.

As an application of this new formulation, we derived the Lagrangian formulation for the nonlinear laser-wakefield equations (38.a and b) in the multi-dimensional quasi-static (MDQS) approximation.[12] In the one-dimensional QS (1DQS) limit,[22] we recovered a Lagrangian formulation previously constructed by Decker and Mori.[13] Next, we applied the Noether method to obtain exact conservation laws for the MDQS nonlinear equations.

In future work, we will turn our attention to applications of the Lagrangian formulations (11) and (33). In particular, we are interested in finding out ways of optimizing the background plasma density profile, for a given laser pulse, to yield an optimal plasma wake field (which gives the necessary accelerating potential). Another possible direction would be to generalize the MDQS equations (38.a-b) to include finite background-density gradients.

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