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A geometric view of Hamiltonian perturbation theory

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Abstract

The variational formulation for Lie-transform Hamiltonian perturbation theory is presented in terms of an action functional defined on a two-dimensional parameter space. A fundamental equation in Hamiltonian perturbation theory is shown to result from the freedom of choice of the integration path for the action functional. © 2001 Elsevier Science B.V. All rights reserved.

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The paradigm of canonical Hamiltonian perturbation theory [1,2] involves the transformation of an *exact* Hamiltonian H_ϵ, which depends continuously on a perturbation parameter ϵ, into a *reference* Hamiltonian H_0 for which the Hamilton equations $\partial_t \mathbf{z} = \{\mathbf{z}, H_0\}$ have a known solution (unless otherwise noted $\mathbf{z}$ denotes canonical phase-space coordinates and $\{\ ,\ \}$ denotes the canonical Poisson bracket). According to the Lie-transform approach to Hamiltonian perturbation theory [1], the transformation $H_\epsilon \to H_0$ is *induced* by a reversible (time-dependent) phase-space transformation from the old phase-space coordinates $\mathbf{z}$ to the new phase-space coordinates $\bar{\mathbf{z}}(\epsilon) = T_\epsilon \mathbf{z}$, where T_ϵ is an operator defined in terms of a generating scalar field S_ϵ as

$$T_\epsilon \equiv \exp\left(\int_0^\epsilon \{S_\sigma,\ \} d\sigma \right).$$

For a time-dependent phase-space transformation $\mathbf{z} \to \bar{\mathbf{z}}(\epsilon)$ generated by the scalar field $S_\epsilon(\mathbf{z}, t)$, the transformation from the old Hamiltonian H_ϵ to the new Hamiltonian H_0 is expressed as [2,3]

$$H_0\big(\bar{\mathbf{z}}(\epsilon), t\big) \equiv H_\epsilon(\mathbf{z}, t) - \int_0^\epsilon \frac{\partial S_\sigma(\mathbf{z}, t)}{\partial t}\, d\sigma. \tag{1}$$

The transformations $\mathbf{z} \to \bar{\mathbf{z}}$ and $H_\epsilon \to H_0$ are therefore completely determined by the perturbed Hamiltonian $H_\epsilon(\mathbf{z}, t)$ and the phase-space generating function $S_\epsilon(\mathbf{z}, t)$. Since H_0 is independent of ϵ by construction (i.e., $\partial_\epsilon H_0 \equiv 0$), the ϵ-derivative of both sides in (1) yields a dynamical evolution (henceforth known as the Lie-transform perturbation equation)

$$\partial_t S + \{S, H\} = \partial_\epsilon H, \tag{2}$$

where the parametric ϵ-dependence is included with the time dependence. The ϵ-perturbed Hamilton equations, on the other hand, are now expressed as

$$\frac{\partial z^\alpha(t,\epsilon)}{\partial t} = \big\{z^\alpha, H(\mathbf{z}; t, \epsilon)\big\} \tag{3}$$

and

$$\frac{\partial z^\alpha(t,\epsilon)}{\partial \epsilon} = \big\{z^\alpha, S(\mathbf{z}; t, \epsilon)\big\}. \tag{4}$$

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Hence, whereas the Hamiltonian H is the generating function for the infinitesimal canonical transformations associated with the dynamical evolution (3) of the Hamiltonian system (henceforth referred to as the t-dynamics), the generating function S is the Hamiltonian for the *perturbation* evolution (4) of the Hamiltonian system (henceforth referred to as the ϵ-dynamics). We note that the order with which the Hamiltonian system is evolved and perturbed should be immaterial, i.e., the same dynamical state $\mathbf{z}(t,\epsilon) \equiv \mathbf{z}_\epsilon(t)$ can be reached by either evolving the unperturbed system first $[\mathbf{z}_0(t=0) \to \mathbf{z}_0(t)]$ and then perturbing it $[\mathbf{z}_0(t) \to \mathbf{z}_\epsilon(t)]$ or perturbing the system first $[\mathbf{z}_0(t=0) \to \mathbf{z}_\epsilon(t=0)]$ and then evolving it $[\mathbf{z}_\epsilon(t=0) \to \mathbf{z}_\epsilon(t)]$.

The purpose of this Letter is to present the variational formulation for the Lie-transform perturbation equation (2) and the multi-Hamilton equations (3) and (4). In particular, we show that the Lie-transform perturbation equation (2) is a direct consequence of path independence in our variational formulation. First, we introduce an extended phase-space Lagrangian Γ on the (t,ϵ)-plane defined as

$$\Gamma\big(\mathbf{Z}(t,\epsilon);t,\epsilon\big) \equiv \mathbf{P}(t,\epsilon)\cdot d\mathbf{Q}(t,\epsilon) - H\big(\mathbf{Z}(t,\epsilon);t,\epsilon\big)\,dt - S\big(\mathbf{Z}(t,\epsilon);t,\epsilon\big)\,d\epsilon, \tag{5}$$

where

$$Z^\alpha : (t,\epsilon) \mapsto Z^\alpha(t,\epsilon) \equiv \big(\mathbf{Q}(t,\epsilon), \mathbf{P}(t,\epsilon)\big) \tag{6}$$

denotes a generic mapping from the (t,ϵ)-plane to the $2N$-dimensional phase space, with

$$dZ^\alpha(t,\epsilon) \equiv \frac{\partial Z^\alpha(t,\epsilon)}{\partial t}\,dt + \frac{\partial Z^\alpha(t,\epsilon)}{\partial \epsilon}\,d\epsilon.$$

(In what follows, the uppercase $\mathbf{Z}$ denotes a generic mapping whereas the lowercase $\mathbf{z}$ denotes a Hamiltonian orbit in phase space.) The one-form (5) is said to be extended in the sense that the ϵ-dynamics Hamiltonian term $-S\,d\epsilon$ has been added to the standard phase-space Lagrangian $\mathbf{P}\cdot d\mathbf{Q} - H\,dt$. Next, we define the action integral

$$\mathcal{A}_C[\mathbf{Z}] \equiv \int_C \Gamma\big(\mathbf{Z}(t,\epsilon);t,\epsilon\big), \tag{7}$$

where C denotes an arbitrary path between two (distinct) points on the two-dimensional (t,ϵ)-plane. For a fixed path C, the action integral $\mathcal{A}_C[\mathbf{Z}]$ is a functional of mapping (6). Holding C fixed, we first consider the variational principle $\delta\mathcal{A}_C[\mathbf{z}] = 0$ corresponding to an arbitrary variation $\delta\mathbf{Z} \equiv \mathbf{Z} - \mathbf{z}$ (which is assumed to vanish at the end points of C). Using Eqs. (5) and (7), we thus find

$$\delta\mathcal{A}_C[\mathbf{z}] = \int_C \delta Z^\alpha\left(\omega_{\alpha\beta}\,dz^\beta - \frac{\partial H}{\partial z^\alpha}\,dt - \frac{\partial S}{\partial z^\alpha}\,d\epsilon\right) + \int_C d(\mathbf{p}\cdot\delta\mathbf{Q}), \tag{8}$$

where the second path integral vanishes since we assumed that $\delta\mathbf{Z}$ vanishes at the end points of C; here, $\omega_{\alpha\beta}$ denotes the components of the canonical Lagrange tensor (i.e., $dp_i \wedge dq^i = 1/2\omega_{\alpha\beta}\,dz^\alpha \wedge dz^\beta$). The multi-Hamilton equations (3) and (4) are automatically recovered in Euler–Lagrange form as $\omega_{\alpha\beta}\,dz^\beta = \partial_\alpha H\,dt + \partial_\alpha S\,d\epsilon$ from the variational principle $\delta\mathcal{A}_C[\mathbf{z}] = 0$ for arbitrary variations $\delta\mathbf{Z}$ and an arbitrary path C.

In the standard variational principle for single-particle Hamiltonian dynamics [3], the action functional involves a time integration of the phase-space Lagrangian $\mathbf{P}\cdot d\mathbf{Q} - H\,dt$ from an initial time t_1 to a final time t_2; the initial and final times play no role and the variational principle yields the standard Hamilton equations. The variational principle based on the action functional (7) presents us with a new problem: determining how the freedom of choice in selecting the path C on the (t,ϵ)-plane is expressed mathematically. To resolve this problem, we first consider two different paths C and C', both having the same end points, and we choose mapping (6) to be a multi-Hamiltonian orbit $\mathbf{z}(t,\epsilon)$, i.e., a solution of Eqs. (3) and (4). Next, according to Stokes' theorem [4], the difference between $\mathcal{A}_C[\mathbf{z}]$ and $\mathcal{A}_{C'}[\mathbf{z}]$ is evaluated as

$$\mathcal{A}_C[\mathbf{z}] - \mathcal{A}_{C'}[\mathbf{z}] = \oint_{C-C'} \Gamma = \int_{\mathcal{D}} d\Gamma, \tag{9}$$

where $\mathcal{D}$ denotes the area in the (t,ϵ)-plane enclosed by the two paths C and C' (i.e., $C - C' \equiv \partial\mathcal{D}$ denotes the contour of $\mathcal{D}$). Using Eqs. (3)–(5), the two-form $d\Gamma$ appearing in Eq. (9) is

$$d\Gamma\big(t,\epsilon;\mathbf{z}(t,\epsilon)\big) = d\epsilon \wedge dt\left(\frac{dS}{dt} - \frac{dH}{d\epsilon} - \{S,H\}\right), \tag{10}$$

where the operators d/dt and $d/d\epsilon$ are defined as

$$d/dt \equiv \partial_t + \{\ ,H\},$$
$$d/d\epsilon \equiv \partial_\epsilon + \{\ ,S\}. \tag{11}$$

From Eq. (9), we see that, for a fixed Hamiltonian orbit $\mathbf{z}(t,\epsilon)$, the condition for the path-independence of the action integral $\mathcal{A}_C[\mathbf{z}]$ (i.e., $\mathcal{A}_C[\mathbf{z}] = \mathcal{A}_{C'}[\mathbf{z}]$) is $d\Gamma = 0$. From Eq. (10), we see that this condition holds provided H and S satisfy the following constraint equation:

$$\frac{\partial S}{\partial t} - \frac{\partial H}{\partial \epsilon} = \{H, S\}, \tag{12}$$

which is exactly the Lie-transform perturbation equation (2) appearing in Lie-transform Hamiltonian perturbation theory. We can also verify by using the operators defined in (11) and the Jacobi identity for the Poisson bracket that constraint (12) ensures that the flows associated with t- and ϵ-dynamics commute:

$$\left[\frac{d}{dt}, \frac{d}{d\epsilon}\right] g(\mathbf{z};t,\epsilon) = \frac{d}{dt}\left(\frac{dg}{d\epsilon}\right) - \frac{d}{d\epsilon}\left(\frac{dg}{dt}\right) = 0 \tag{13}$$

for all $g(\mathbf{z};t,\epsilon)$. As expected, the interpretation of the commutation relation (13) is that the order with which the Hamiltonian system is evolved (under the t-dynamics) and is perturbed (under the ϵ-dynamics) is indeed immaterial.

So far we have presented the variational formulation of Lie-transform canonical Hamiltonian perturbation theory. Most recent applications of Hamiltonian perturbation theory in plasma physics are carried out using non-canonical phase-space coordinates [5,6]. Non-canonical Lie-transform Hamiltonian perturbation theory possesses a variational formulation expressed in terms of the extended non-canonical phase-space Lagrangian

$$\Gamma = \left[m\mathbf{v} + \frac{e}{c}\mathbf{A}(\mathbf{x};t,\epsilon)\right] \cdot d\mathbf{x} - H(\mathbf{x},\mathbf{v};t,\epsilon)\,dt - S(\mathbf{x},\mathbf{v};t,\epsilon)\,d\epsilon, \tag{14}$$

where the non-canonical coordinates $(\mathbf{x},\mathbf{v})$ denote the particle position and its velocity while $\mathbf{A}(\mathbf{x};t,\epsilon)$ denotes the perturbed magnetic vector potential. For a fixed path C in the (t,ϵ)-plane, the variational principle $\delta(\int_C \Gamma) = 0$ yields the Euler–Lagrange equations

$$m\,d\mathbf{x} = \frac{\partial H}{\partial \mathbf{v}}\,dt + \frac{\partial S}{\partial \mathbf{v}}\,d\epsilon,$$

$$m\,d\mathbf{v} = -\left(\nabla H + \frac{e}{c}\frac{\partial \mathbf{A}}{\partial t}\right)dt - \left(\nabla S + \frac{e}{c}\frac{\partial \mathbf{A}}{\partial \epsilon}\right)d\epsilon + \frac{e}{c}\,d\mathbf{x} \times \mathbf{B},$$

which can be written in multi-Hamilton form as

$$\partial_t z^\alpha = \{z^\alpha, H\}_{\mathrm{nc}} + (e/c)\partial_t \mathbf{A} \cdot \{\mathbf{x}, z^\alpha\}_{\mathrm{nc}},$$
$$\partial_\epsilon z^\alpha = \{z^\alpha, S\}_{\mathrm{nc}} + (e/c)\partial_\epsilon \mathbf{A} \cdot \{\mathbf{x}, z^\alpha\}_{\mathrm{nc}}, \tag{15}$$

where $\{\ ,\ \}_{\mathrm{nc}}$ denotes the non-canonical Poisson bracket

$$\{a,b\}_{\mathrm{nc}} \equiv \frac{1}{m}\left(\nabla a \cdot \frac{\partial b}{\partial \mathbf{v}} - \frac{\partial a}{\partial \mathbf{v}} \cdot \nabla b\right) + \frac{e\mathbf{B}}{m^2 c} \cdot \left(\frac{\partial a}{\partial \mathbf{v}} \times \frac{\partial b}{\partial \mathbf{v}}\right).$$

The condition $d\Gamma = 0$ for path independence, on the other hand, yields the Lie-transform non-canonical perturbation equation

$$\frac{\partial S}{\partial t} - \frac{\partial H}{\partial \epsilon} = \{H, S\}_{\mathrm{nc}} + \frac{e}{mc}\left(\frac{\partial \mathbf{A}}{\partial t} \cdot \frac{\partial S}{\partial \mathbf{v}} - \frac{\partial \mathbf{A}}{\partial \epsilon} \cdot \frac{\partial H}{\partial \mathbf{v}}\right). \tag{16}$$

This non-canonical Hamiltonian perturbation equation plays a prominent role in the derivation of the reduced non-linear gyrokinetic equations describing the perturbed Hamiltonian dynamics of charged particles under the influence of low-frequency electromagnetic fluctuations in a magnetized plasma [6]. A linearized version of (16) also plays a prominent role in the free-energy method developed by Morrison and Pfirsch [7–9] to investigate the linear stability of various plasma equilibria.

Finally, we note that the variational formulation for Lie-transform Hamiltonian perturbation theory can be generalized to include several perturbation parameters. Consider the $(k+1)$-component perturbation vector $\boldsymbol{\epsilon} \equiv (\epsilon^0, \epsilon^1, \ldots, \epsilon^k)$, where each perturbation parameter ϵ^a is paired with a Hamiltonian S_a (with $\epsilon^0 \equiv t$ and $S_0 \equiv H$). Hence, the multi-Hamilton equations (3) and (4) are now replaced by the set of multi-Hamilton equations

$$\frac{\partial z^\alpha(\boldsymbol{\epsilon})}{\partial \epsilon^a} = \left\{z^\alpha, S_a(\mathbf{z};\boldsymbol{\epsilon})\right\}, \tag{17}$$

where $a = 0,1,\ldots,k$. We now define the action functional $\mathcal{A}_C[\mathbf{Z}] \equiv \int_C \Gamma(\mathbf{Z}(\boldsymbol{\epsilon});\boldsymbol{\epsilon})$ in terms of a path

C in the $(k+1)$-dimensional $\boldsymbol{\epsilon}$-space and the one-form,

$$\Gamma\big(\mathbf{Z}(\boldsymbol{\epsilon});\boldsymbol{\epsilon}\big) \equiv \mathbf{P}(\boldsymbol{\epsilon})\cdot d\mathbf{Q}(\boldsymbol{\epsilon}) - S_a\big(\mathbf{Z}(\boldsymbol{\epsilon});\boldsymbol{\epsilon}\big)\,d\epsilon^a, \qquad (18)$$

where $\mathbf{Z}:\boldsymbol{\epsilon}\mapsto\mathbf{Z}(\boldsymbol{\epsilon})$ denotes a generic mapping from $\boldsymbol{\epsilon}$-space to phase space. For a fixed (but arbitrary) path C, we recover the multi-Hamilton equations (17) from the variational principle $\delta\mathcal{A}_C[\mathbf{z}]=0$. In addition, for a fixed multi-Hamiltonian orbit $\mathbf{z}(\boldsymbol{\epsilon})=(\mathbf{q}(\boldsymbol{\epsilon}),\mathbf{p}(\boldsymbol{\epsilon}))$, the $(k+1)$ equations corresponding to the condition for the path-independence of $\mathcal{A}_C[\mathbf{z}]$ (i.e., $d\Gamma=0$) are

$$\partial_a S_b - \partial_b S_a = \{S_a, S_b\}. \qquad (19)$$

Note that this set of equations is invariant under the gauge transformation $S_a \to S_a - \partial\chi/\partial\epsilon^a$, where $\chi(\mathbf{q};\boldsymbol{\epsilon})$ is an arbitrary scalar field. Gauge invariance of the multi-Hamiltonian dynamics requires that the phase-space Lagrangian Γ transform as $\Gamma\to\Gamma+d\chi$ and $\mathbf{p}\to\mathbf{p}+\partial\chi/\partial\mathbf{q}$. The gauge properties of the generating function S are further elucidated in Ref. [10].

The work presented here introduced a variational formulation for Lie-transform Hamiltonian perturbation theory. This work follows recent developments in the variational formulation of exact and reduced Vlasov–Maxwell equations [10–13] as well as other important dynamical field equations in plasma physics (e.g., kinetic-MHD equations [10] and drift-wave equations [14]).

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