Variational principle for nonlinear gyrokinetic Vlasov–Maxwell equations

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(Received 30 June 2000; accepted 8 September 2000)

A new variational principle for the nonlinear gyrokinetic Vlasov–Maxwell equations is presented. This *Eulerian* variational principle uses *constrained* variations for the gyrocenter Vlasov distribution in eight-dimensional extended phase space and turns out to be simpler than the *Lagrangian* variational principle recently presented by H. Sugama [Phys. Plasmas **7**, 466 (2000)]. A *local* energy conservation law is then derived explicitly by the Noether method. In future work, this new variational principle will be used to derive self-consistent, nonlinear, low-frequency Vlasov–Maxwell bounce-gyrokinetic equations, in which the fast gyromotion and bounce-motion time scales have been eliminated. © *2000 American Institute of Physics.* [S1070-664X(00)04412-7]

I. INTRODUCTION

Over the past two decades, much attention has been paid to the systematic derivation of *reduced* Hamiltonian dynamics and their associated *reduced* Vlasov kinetic descriptions in which fast orbital time scales have been asymptotically eliminated.[1–4] Thus for numerical particle-kinetic simulations of magnetized plasmas,[5] great advantages have been obtained by solving a reduced Vlasov equation in a five-dimensional space instead of solving the exact Vlasov equation in six-dimensional phase-space.[6]

When a self-consistent treatment of particles and fields is required, the charged-particle densities and current densities in the Maxwell equations need to be expressed in terms of moments of the reduced Vlasov distribution.[4] In contrast to the derivation of reduced single-particle Hamiltonian dynamics, the derivation of self-consistent reduced Maxwell equations has generally proceeded in a less systematic ad hoc fashion. In the present work, a more systematic procedure for the derivation of self-consistent reduced Vlasov–Maxwell equations is presented based on a new Eulerian variational principle[7] for the exact Vlasov–Maxwell equations.

A. Eulerian variational principle for exact Vlasov–Maxwell equations

The variational formulation for the Vlasov–Maxwell equations has been a topic of interest in plasma physics ever since Low presented his *Lagrangian* variational principle,[8] which combined the standard action functional for the electromagnetic field with an action functional for *particles*. Since then a variety of variational formulations for the Vlasov–Maxwell equations have appeared,[9,10] each one more or less based on the low Lagrangian formalism.

A new Eulerian variational principle for the Vlasov–Maxwell equations, which is simpler than all previous variational formulations, was recently presented in Ref. 7. Whereas the Eulerian variational principle of Cendra *et al.*[10] considered constrained variations on the particle dynamics and the Vlasov distribution expressed in terms of a *six-dimensional* virtual displacement vector field **w**, our new variational principle considers constrained variations of the Vlasov distribution itself on *eight-dimensional* extended phase space expressed in terms of the Poisson bracket on this extended phase space and a single scalar field $\mathcal{S}$ which generates a virtual displacement in extended phase space.

The new variational principle for the Vlasov–Maxwell equations is expressed as[7]

$$\delta\mathcal{A}[\mathcal{F},A^{\mu}] \equiv \delta(\mathcal{A}_V[\mathcal{F},A^{\mu}] + \mathcal{A}_M[A^{\mu}]) = 0, \tag{1}$$

where the action functional $\mathcal{A}$ is divided into a Vlasov (V) part, denoted $\mathcal{A}_V$, and a Maxwell (M) part, denoted $\mathcal{A}_M$. The Vlasov action functional has the simple form[7]

$$\mathcal{A}_V[\mathcal{F},A^{\mu}] \equiv -\int d^8\mathcal{Z}\,\mathcal{F}(\mathcal{Z})\,\mathcal{H}(\mathcal{Z}), \tag{2}$$

where $\mathcal{F}(\mathcal{Z})$ denotes the Vlasov distribution on an eight-dimensional extended phase space with coordinates $\mathcal{Z} \equiv (\mathbf{z};w,t)$ (here, $\mathbf{z}$ denotes six-dimensional phase-space coordinates and the energy coordinate w is conjugate to time t), and $\mathcal{H}(\mathcal{Z}) \equiv H(\mathbf{z},t) - w$ denotes the Hamiltonian on this extended phase space. The Maxwell action functional, on the other hand, is given by the standard action functional for the electromagnetic field,

$$\mathcal{A}_M[A^{\mu}] \equiv \int d^4x \frac{1}{16\pi} \mathsf{F}_{\mu\nu}\mathsf{F}^{\nu\mu}, \tag{3}$$

where the field tensor F is expressed in terms of the four-potential $A^{\mu} = (\phi,\mathbf{A})$ as $\mathsf{F}_{\mu\nu} \equiv \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$. The variational principle (1) is said to be *Eulerian* since variations are evaluated at a fixed point $\mathcal{Z}$ in extended phase space for $\delta\mathcal{F}(\mathcal{Z})$ or at a fixed point $x = (ct,\mathbf{x})$ in four-dimensional space-time for $\delta A^{\mu}(x)$.

Under general variations $\delta\mathcal{F}$ of the extended phase-space Vlasov distribution $\mathcal{F}$, the variational principle (1) yields

$$\frac{\delta\mathcal{A}_V}{\delta\mathcal{F}} \equiv -\mathcal{H} = 0, \tag{4}$$

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i.e., physical motion in extended phase space takes place on the surface $\mathcal{H}(\mathcal{Z})=0$ or $w=H(\mathbf{z},t)$. By imposing this physical condition on the extended phase-space Vlasov distribution $\mathcal{F}(\mathcal{Z})$, we find that it should be of the form

$$\mathcal{F}(\mathcal{Z})\equiv\delta[w-H(\mathbf{z},t)]f(\mathbf{z},t), \tag{5}$$

where $f(\mathbf{z},t)$ is the time-dependent Vlasov distribution on six-dimensional phase space. Instead of general variations $\delta\mathcal{F}$, the present Eulerian variational principle considers *constrained* variations of the form

$$\delta\mathcal{F}\equiv\{\mathcal{S},\mathcal{F}\}_{\mathcal{Z}}, \tag{6}$$

where $\{\ ,\ \}_{\mathcal{Z}}$ is the Poisson bracket on extended phase space and $\mathcal{S}$ is the generating function for an infinitesimal transformation on extended phase space (i.e., $\mathcal{Z}\rightarrow\mathcal{Z}+\delta\mathcal{Z}$, where $\delta\mathcal{Z}^{\alpha}=\{\mathcal{Z}^{\alpha},\mathcal{S}\}_{\mathcal{Z}}$). Note that (6) can also be written using canonical coordinates as $\delta\mathcal{F}\equiv-\partial_{\alpha}(\delta\mathcal{Z}^{\alpha}\mathcal{F})$, which is analogous in form to the expression $\delta n\equiv-\nabla\cdot(\boldsymbol{\xi}n)$ for the constrained variation of fluid density n in ideal magnetohydrodynamics. Thus under *constrained* variations of the Vlasov distribution $\mathcal{F}$, the variational principle (1) yields the Vlasov equation in extended phase space,

$$\{\mathcal{F},\mathcal{H}\}_{\mathcal{Z}}\equiv0. \tag{7}$$

Substituting (5) into the extended Vlasov equation (7) yields the Vlasov equation in phase space,

$$\frac{\partial f(\mathbf{z},t)}{\partial t}+\{f(\mathbf{z},t),H(\mathbf{z},t)\}=0, \tag{8}$$

where $\{\ ,\ \}$ is the Poisson bracket on phase space. Lastly, under general variations δA^{ν} of the electromagnetic four-potential A^{ν}, the variational principle (1) yields the Maxwell equations

$$-\frac{1}{4\pi}\frac{\partial}{\partial x^{\mu}}F^{\mu\nu}(x)=\frac{\delta\mathcal{A}_{V}}{\delta A_{\nu}(x)}, \tag{9}$$

where the right-hand side represents the self-consistent particle current-density four-vector expressed as the functional derivative of the Vlasov action functional $\mathcal{A}_{V}[\mathcal{F},A^{\nu}]$ with respect to the four-potential $A^{\nu}(x)$.

B. Eulerian variational principle for reduced Vlasov–Maxwell equations

The most efficient method for deriving reduced Hamilton and Vlasov kinetic equations is based on Hamiltonian[1] and phase-space Lagrangian[11] Lie-perturbation methods. In general Lie-perturbation theory,[12] the asymptotic elimination of a fast time scale (represented by the angle θ) proceeds by a time-dependent near-identity phase-space transformation

$$\mathcal{Z}\rightarrow\overline{\mathcal{Z}}(\mathcal{Z},\epsilon)\equiv T_{\epsilon}\mathcal{Z}, \tag{10}$$

where ϵ denotes a dimensionless perturbation parameter. Here, the near-identity transformation is explicitly expressed in terms of generating vector fields $(\mathcal{G}_1,\mathcal{G}_2,\ldots)$,

$$\overline{\mathcal{Z}}^{\alpha}(\mathcal{Z},\epsilon)=\mathcal{Z}^{\alpha}+\epsilon\mathcal{G}_1^{\alpha}+\epsilon^2\left(\mathcal{G}_2^{\alpha}+\frac{1}{2\mathcal{G}_1^{\beta}}\frac{\partial\mathcal{G}_1^{\alpha}}{\partial\mathcal{Z}^{\beta}}\right)+\cdots, \tag{11}$$

where the nth-order generating vector field $\mathcal{G}_n$ is chosen to remove the fast time scale at order ϵ^n in the Hamiltonian dynamics. As a result, the adiabatic invariant $\overline{J}$ (the action variable conjugate to the fast angle $\overline{\theta}$) is constructed as an asymptotic expansion in powers of ϵ with the following property; if the perturbation analysis is performed up to ϵ^n, then $d\overline{J}/dt\equiv-\partial\overline{H}/\partial\overline{\theta}=\mathcal{O}(\epsilon^{n+1})$, i.e., $\overline{H}=\overline{H}_0+\epsilon\overline{H}_1+\cdots+\epsilon^n\overline{H}_n$ is independent of the fast angle $\overline{\theta}$.

Under the near-identity phase-space transformation (10), the Vlasov action functional (2) becomes the *reduced* Vlasov action functional

$$\mathcal{A}_{VR}[\overline{\mathcal{F}},A^{\mu}]=-\int d^8\overline{\mathcal{Z}}\overline{\mathcal{F}}(\overline{\mathcal{Z}})\overline{\mathcal{H}}(\overline{\mathcal{Z}}), \tag{12}$$

where $\overline{\mathcal{F}}(\overline{\mathcal{Z}})\equiv(T_{\epsilon}^{*})^{-1}\mathcal{F}(\overline{\mathcal{Z}})\equiv\mathcal{F}(\mathcal{Z})$ denotes the *reduced* Vlasov distribution expressed as the *pull-back* of the old Vlasov distribution $\mathcal{F}$. Here, the pull-back operator is defined as

$$(T_{\epsilon}^{*})^{-1}\mathcal{F}(\overline{\mathcal{Z}})\equiv\mathcal{F}(\overline{\mathcal{Z}})-\epsilon\mathcal{G}_1^{\alpha}(\overline{\mathcal{Z}})\frac{\partial\mathcal{F}}{\partial\overline{\mathcal{Z}}^{\alpha}}+\cdots. \tag{13}$$

Note that by construction the reduced Hamiltonian

$$\overline{\mathcal{H}}(\overline{\mathcal{Z}})\equiv(T_{\epsilon}^{*})^{-1}\mathcal{H}(\overline{\mathcal{Z}})=(\overline{H}_0-\overline{w})+\epsilon\overline{H}_1+\epsilon^2\overline{H}_2+\cdots, \tag{14}$$

and the reduced Vlasov distribution $\overline{\mathcal{F}}$ are independent of the fast time scale. By applying the variational principle $\delta\mathcal{A}_{VR}=0$ with the constrained variation $\delta\overline{\mathcal{F}}\equiv\{\overline{\mathcal{S}},\overline{\mathcal{F}}\}_{\overline{\mathcal{Z}}}$, it is a simple task to derive the *reduced* Vlasov equation,

$$\{\overline{\mathcal{F}},\overline{\mathcal{H}}\}_{\overline{\mathcal{Z}}}=0, \tag{15}$$

where $\{\ ,\ \}_{\overline{\mathcal{Z}}}$ is the Poisson bracket on the transformed extended phase space. We note that in general the pull-back operator (13) depends explicitly on the electromagnetic potentials $(\phi,\mathbf{A})$ and thus the variation of the reduced Vlasov functional (12) with respect to ϕ and $\mathbf{A}$ yields the correct particle densities and currents expressed in terms of moments of the reduced Vlasov distribution $\overline{\mathcal{F}}$ (i.e., the reduced particle currents are calculated from the functional derivative $\delta\mathcal{A}_{VR}/\delta A_{\nu}$).

As an explicit application of the variational principle for the reduced Vlasov–Maxwell equations, we consider the nonlinear low-frequency gyrokinetic Vlasov–Maxwell equations.[4] These equations are obtained through a sequence of two near-identity phase-space transformations: a time-independent guiding-center transformation and a time-dependent gyrocenter transformation. The Eulerian variational principle presented below turns out to be simpler than the Lagrangian variational principle presented recently by Sugama,[13] in which an action functional for gyrocenter particles was derived from the Low Lagrangian formalism.

The remainder of this paper is organized as follows. In Sec. II, the derivation of the nonlinear gyrocenter Hamiltonian dynamics is summarized (following work presented earlier[4]). In Sec. III, the Eulerian variational principle for the nonlinear low-frequency gyrokinetic Vlasov–Maxwell equations is presented; the present work is a nonlinear generalization of earlier work.[14] In Sec. IV, the Noether method is used to obtain an explicit expression for the *local* energy conservation law for the low-frequency gyrokinetic Vlasov–Maxwell equations; the standard *global* gyrokinetic energy conservation law[15] is recovered. Lastly, in Sec. V, our work is summarized and future work is discussed.

II. NONLINEAR LOW-FREQUENCY GYROCENTER PHASE-SPACE TRANSFORMATION

In this section, we consider the following situation: A background magnetized plasma represented by time-independent electromagnetic potentials $A_0^\mu=(\phi_0,\mathbf{A}_0)$ is perturbed with low-frequency electromagnetic potentials $A_1^\mu=(\phi_1,\mathbf{A}_1)$. The amplitude of these perturbations is ordered with a dimensionless parameter $\epsilon\ll 1$, so that the exact electromagnetic field tensor becomes $\mathsf{F}=\mathsf{F}_0+\epsilon\mathsf{F}_1$.

The exact dynamics of a particle of mass m and charge e is described in eight-dimensional extended phase space in terms of the extended phase-space Lagrangian $\gamma=\gamma_0+\epsilon\gamma_1$, where $\gamma_0\equiv[(e/c)\mathbf{A}_0+m\mathbf{p}]\cdot d\mathbf{x}-w\,dt$ and $\gamma_1\equiv(e/c)\mathbf{A}_1$, and the extended phase-space Hamiltonian $\mathcal{H}=\mathcal{H}_0+\epsilon\mathcal{H}_1$, where $\mathcal{H}_0\equiv|\mathbf{p}|^2/2m+e\phi_0-w$ and $\mathcal{H}_1\equiv e\phi_1$. The Poisson bracket $\{\ ,\ \}_{\mathcal{Z}}$ on extended phase space is obtained from the extended phase-space Lagrangian γ by standard means.[12]

When a magnetized plasma is perturbed with low-frequency electromagnetic fluctuations, it is possible to construct a reduced dynamical description by asymptotically eliminating the fast gyromotion orbital time scale. The standard low-frequency gyrokinetic analysis[3,4] proceeds by a sequence of two near-identity phase-space transformations: a time-independent *guiding-center* phase-space transformation and a time-dependent *gyrocenter* phase-space transformation.

A. Time-independent guiding-center extended-phase-space transformation

The first near-identity transformation is the *guiding-center* phase-space transformation, with a small dimensionless parameter $\epsilon_B\equiv\rho/L_B\ll 1$ defined as the ratio of the characteristic gyroradius ρ (for a charged particle of mass m and charge e) and the background magnetic-field length scale L_B. This transformation is designed to remove the fast gyromotion time scale associated with the time-independent background magnetic field $\mathbf{B}_0=\nabla\times\mathbf{A}_0$ associated with an unperturbed magnetized plasma.

In previous work,[4] this transformation was carried out to second order in ϵ_B with the scalar potential ϕ_0 ordered at zeroth order in ϵ_B. The results of the guiding-center analysis presented in Ref. 4 are summarized as follows. First, the guiding-center transformation yields the following guiding-center coordinates $(\mathbf{R},p_\parallel,\mu,\theta,w,t)\equiv\mathcal{Z}$, where $\mathbf{R}$ is the guiding-center position, $p_\parallel\equiv mv_\parallel$ is the guiding-center momentum parallel to the magnetic field, μ is the guiding-center magnetic moment, θ is the gyroangle, and (w,t) are the canonically conjugate guiding-center energy-time coordinates. Next, the unperturbed guiding-center extended phase-space Lagrangian is

$$\gamma_{gc}\equiv\frac{e}{c}\mathbf{A}^*\cdot d\mathbf{R}+\mu(mc/e)\,d\theta-w\,dt, \tag{16}$$

where $\mathbf{A}^*\equiv\mathbf{A}_0+(cp_\parallel/e)\hat{\mathbf{b}}$ is the effective unperturbed vector potential, with $\hat{\mathbf{b}}\equiv\mathbf{B}_0/B_0$; we henceforth omit displaying the dimensionless guiding-center parameter ϵ_B for simplicity. The unperturbed extended phase-space guiding-center Hamiltonian is[4]

$$\mathcal{H}_{gc}=e\phi_0+\frac{p_\parallel^2}{2m}+\mu B_0-w\equiv H_{gc}-w. \tag{17}$$

Lastly, from the unperturbed guiding-center phase-space Lagrangian (16), we obtain the unperturbed guiding-center Poisson bracket $\{\ ,\ \}$, given here in terms of two arbitrary functions $\mathcal{F}$ and $\mathcal{G}$ on extended guiding-center phase space as[4]

$$\begin{aligned}\{\mathcal{F},\mathcal{G}\}_{\mathcal{Z}}\equiv{}&\frac{e}{mc}\left(\frac{\partial\mathcal{F}}{\partial\theta}\frac{\partial\mathcal{G}}{\partial\mu}-\frac{\partial\mathcal{F}}{\partial\mu}\frac{\partial\mathcal{G}}{\partial\theta}\right)\\&+\frac{\mathbf{B}^*}{B_\parallel^*}\cdot\left(\nabla\mathcal{F}\frac{\partial\mathcal{G}}{\partial p_\parallel}-\frac{\partial\mathcal{F}}{\partial p_\parallel}\nabla\mathcal{G}\right)-\frac{c\hat{\mathbf{b}}}{eB_\parallel^*}\cdot\nabla\mathcal{F}\times\nabla\mathcal{G}\\&+\left(\frac{\partial\mathcal{F}}{\partial w}\frac{\partial\mathcal{G}}{\partial t}-\frac{\partial\mathcal{F}}{\partial t}\frac{\partial\mathcal{G}}{\partial w}\right),\end{aligned} \tag{18}$$

where $\mathbf{B}^*\equiv\nabla\times\mathbf{A}^*$ and $B_\parallel^*\equiv\hat{\mathbf{b}}\cdot\mathbf{B}^*$. The guiding-center Hamiltonian dynamics in the local reference moving frame is thus expressed in terms of the Hamiltonian (17) and the Poisson bracket (18) as $\dot{\mathcal{Z}}^\alpha\equiv\{\mathcal{Z}^\alpha,\mathcal{H}_{gc}\}_{\mathcal{Z}}$. In particular, we have $\dot{\mu}\equiv 0$ since H_{gc} is independent of the fast gyroangle θ (to all orders in ϵ_B).

B. Time-dependent gyrocenter extended-phase-space transformation

We now consider how guiding-center Hamiltonian dynamics is affected by the introduction of low-frequency electromagnetic field fluctuations $(\phi_1,\mathbf{A}_1)$. These fluctuations are assumed to satisfy the low-frequency gyrokinetic ordering[3,4]

$$\begin{aligned}&\omega/\Omega_0\equiv\epsilon_\omega\ll 1,\\&k_\parallel/|\mathbf{k}_\perp|=\mathcal{O}(\epsilon_\omega),\\&|\mathbf{k}_\perp|\rho=\mathcal{O}(1),\end{aligned} \tag{19}$$

where $\Omega_0\equiv eB_0/mc$ denotes the charged-particle's gyrofrequency and ϵ_ω is a small dimensionless ordering parameter associated with the electromagnetic perturbations space–time scales, with ω the characteristic wave frequency, $k_\parallel$ the characteristic parallel wavenumber and $\mathbf{k}_\perp$ the characteristic perpendicular wave vector (both with respect to the unperturbed magnetic field $\mathbf{B}_0$).

Under these electromagnetic perturbations, the guiding-center phase-space Lagrangian (16) and Hamiltonian (17) become

$$\gamma_{gc}'\equiv\gamma_{gc0}+\epsilon\gamma_{gc1}\quad\text{and}\quad\mathcal{H}_{gc}'\equiv\mathcal{H}_{gc0}+\epsilon\mathcal{H}_{gc1}, \tag{20}$$

where ϵ is a dimensionless ordering parameter associated with the amplitude of the electromagnetic perturbation potentials $(\phi_1, \mathbf{A}_1)$ and the zeroth-order guiding-center phase-space Lagrangian γ_{gc0} and Hamiltonian $\mathcal{H}_{gc0}$ are given by (16) and (17), respectively. The first-order guiding-center phase-space Lagrangian γ_{gc1} and Hamiltonian $\mathcal{H}_{gc1}$, on the other hand, are

$$\begin{aligned}
\gamma_{gc1} &= (e/c)\mathbf{A}_1(\mathbf{R}+\boldsymbol{\rho},t)\cdot d(\mathbf{R}+\boldsymbol{\rho}) \\
&\equiv (e/c)\mathbf{A}_{1gc}(\mathbf{R},t;\mu,\theta)\cdot d(\mathbf{R}+\boldsymbol{\rho}) \\
\mathcal{H}_{gc1} &= e\phi_1(\mathbf{R}+\boldsymbol{\rho},t) \\
&\equiv e\phi_{gc}(\mathbf{R},t;\mu,\theta),
\end{aligned} \tag{21}$$

where $\mathbf{A}_{1gc}(\mathbf{R},t;\mu,\theta)$ and $\phi_{1gc}(\mathbf{R},t;\mu,\theta)$ denote perturbation potentials evaluated at a particle's position $\mathbf{x}\equiv\mathbf{R}+\boldsymbol{\rho}$ expressed in terms of the guiding-center position $\mathbf{R}$ and the gyroangle-dependent gyroradius vector $\boldsymbol{\rho}(\mu,\theta)$ (to lowest order in ϵ_B, we ignore the spatial dependence of $\boldsymbol{\rho}$.)

Because of the gyroangle-dependence in the guiding-center perturbation potentials ϕ_{1gc} and $\mathbf{A}_{1gc}$, the guiding-center magnetic moment μ is no longer conserved by the perturbed guiding-center equations of motion, i.e., $\dot{\mu} = \mathcal{O}(\epsilon)$. To remove the gyroangle-dependence from the perturbed guiding-center phase-space Lagrangian and Hamiltonian (20), we proceed by using the time-dependent *gyrocenter* phase-space transformation,

$$\mathcal{Z}\equiv(\mathbf{R},p_\parallel,\mu,\theta,w,t)\rightarrow\bar{\mathcal{Z}}\equiv(\bar{\mathbf{R}},\bar{p}_\parallel,\bar{\mu},\bar{\theta},\bar{w},t),$$

where $\bar{\mathcal{Z}}$ denote *gyrocenter* (gy) extended phase-space coordinates; we note that the time coordinate t is not affected by this transformation.

The results of the nonlinear Hamiltonian gyrocenter perturbation analysis are summarized as follows.[4] To first order in ϵ and zeroth order in ϵ_ω and ϵ_B, this transformation is represented in terms of generating vector fields $(\mathcal{G}_{\rm gy1},\mathcal{G}_{\rm gy2},\ldots)$ as

$$\bar{\mathcal{Z}}^\alpha\equiv\mathcal{Z}^\alpha+\epsilon\mathcal{G}_{\rm gy1}^\alpha+\cdots. \tag{22}$$

The components of the first-order gyrocenter generating vector field $\mathcal{G}_{\rm gy1}$ are

$$\mathcal{G}_{\rm gy1}^\alpha\equiv\{S_{\rm gy1},\mathcal{Z}^\alpha\}_{\mathcal{Z}}+\frac{e}{c}\mathbf{A}_{1gc}\cdot\{\mathbf{R}+\boldsymbol{\rho},\mathcal{Z}^\alpha\}_{\mathcal{Z}}, \tag{23}$$

where $S_{\rm gy1}$ is a gauge function. Through this transformation, the perturbed guiding-center phase-space Lagrangian (20) transforms into the gyrocenter phase-space Lagrangian

$$\gamma_{\rm gy}\equiv\frac{e}{c}\mathbf{A}^*\cdot d\bar{\mathbf{R}}+(mc/e)\bar{\mu}d\bar{\theta}-\bar{w}dt, \tag{24}$$

which is identical in form to the unperturbed guiding-center phase-space Lagrangian (16) so that the gyrocenter Poisson bracket $\{\ ,\ \}_{\bar{\mathcal{Z}}}\equiv\{\ ,\ \}_{\mathcal{Z}}$ has the same form as (18).[16] Up to second order in ϵ, the extended phase-space gyrocenter Hamiltonian is

$$\mathcal{H}_{\rm gy}=(H_{\rm gy0}+\epsilon H_{\rm gy1}+\epsilon^2 H_{\rm gy2})-\bar{w}\equiv H_{\rm gy}-\bar{w}, \tag{25}$$

where $H_{\rm gy}$ is the nonlinear gyrocenter Hamiltonian, given to lowest order in $\epsilon^2\epsilon_\omega$ and $\epsilon^2\epsilon_B$ as

$$\begin{aligned}
H_{\rm gy} &= H_{\rm gy0}+\epsilon e\langle\psi_{1gc}\rangle \\
&\quad+\frac{\epsilon^2}{2}\left(\frac{e^2}{mc^2}\langle|\mathbf{A}_{1gc}|^2\rangle-\frac{e^2}{\Omega_0}\langle\{\tilde{\Psi}_{1gc},\tilde{\psi}_{1gc}\}\rangle\right).
\end{aligned} \tag{26}$$

Here, $\psi_{1gc}\equiv\phi_{1gc}-\mathbf{A}_{1gc}\cdot\mathbf{v}/c$ defines an effective first-order perturbation potential and $\langle\ \rangle$ denotes averaging with respect to the gyroangle $\bar{\theta}$. In (23), the first-order gauge function $S_{\rm gy1}$ is (to lowest in the low-frequency gyrokinetic ordering)

$$S_{\rm gy1}\equiv\frac{e}{\Omega_0}\int\tilde{\psi}_{1gc}d\bar{\theta}\equiv\frac{e}{\Omega_0}\tilde{\Psi}_{1gc}, \tag{27}$$

where $\tilde{\psi}_{1gc}\equiv\psi_{1gc}-\langle\psi_{1gc}\rangle$ denotes the gyroangle-dependent part of ψ_{1gc}.

Lastly, the gyrocenter *pull-back* $(T_{\rm gy}^*)^{-1}$ of any function $\mathcal{F}$ on guiding-center phase space yields a new function $\bar{\mathcal{F}}\equiv(T_{\rm gy}^*)^{-1}\mathcal{F}$ on gyrocenter phase space defined by the relation

$$\bar{\mathcal{F}}(\bar{\mathcal{Z}})\equiv(T_{\rm gy}^*)^{-1}\mathcal{F}(\bar{\mathcal{Z}})\equiv\mathcal{F}(T_{\rm gy}^{-1}\bar{\mathcal{Z}})\equiv\mathcal{F}(\mathcal{Z}). \tag{28}$$

To first order in ϵ, the low-frequency gyrocenter pull-back operator is

$$\begin{aligned}
(T_{\rm gy}^*)^{-1}\mathcal{F} &= \mathcal{F}-\epsilon\left(\frac{e}{\Omega_0}\{\tilde{\Psi}_{1gc},\mathcal{F}\}_{\bar{\mathcal{Z}}}\right. \\
&\quad\left.+\frac{e}{c}\mathbf{A}_{1gc}\cdot\{\bar{\mathbf{R}}+\bar{\boldsymbol{\rho}},\mathcal{F}\}_{\bar{\mathcal{Z}}}\right)+\mathcal{O}(\epsilon^2).
\end{aligned} \tag{29}$$

Note that the extended gyrocenter Hamiltonian $\mathcal{H}_{gy}$ can also be expressed in terms of the gyrocenter pull-back as $\mathcal{H}_{\rm gy}\equiv(T_{\rm gy}^*)^{-1}\mathcal{H}_{gc}$.

III. VARIATIONAL PRINCIPLE FOR NONLINEAR LOW-FREQUENCY GYROKINETIC VLASOV–MAXWELL EQUATIONS

Since time is not affected by the gyrocenter extended phase-space transformation, we henceforth omit displaying its explicit dependence. Moreover, we omit the overbar and subscript gy to denote gyrocenter coordinates and functions on gyrocenter phase space.

The reduced action functional for the nonlinear gyrokinetic Vlasov–Maxwell equations is

$$\mathcal{A}_R=-\int d^8\mathcal{Z}\mathcal{F}(\mathcal{Z})\mathcal{H}(\mathcal{Z})+\int\frac{d^4x}{8\pi}(|\nabla\Phi|^2-|\mathbf{B}|^2), \tag{30}$$

where we henceforth use the notation

$$\Phi\equiv\phi_0+\epsilon\phi_1 \quad\text{and}\quad \mathbf{B}\equiv\mathbf{B}_0+\epsilon\nabla\times\mathbf{A}_1.$$

The absence of the inductive part $-c^{-1}\partial_t\mathbf{A}_1$ of the perturbed electric field $\mathbf{E}_1$ in the Maxwell part of the reduced action functional (30) means that the inductive current $\partial_t\mathbf{E}_1$ will be absent from the Ampere equation; this is consistent with the low-frequency approximation used in nonlinear gyrokinetic theory.

The variational principle $\delta\mathcal{A}_R \equiv \delta\int\mathcal{L}_R d^4x \equiv 0$ for the nonlinear low-frequency gyrokinetic Vlasov–Maxwell equations is based on constrained variations for $\mathcal{F}(\mathcal{Z})$ while variations of the electromagnetic potentials $(\phi,\mathbf{A})$ are *restricted* to variations of the perturbation potentials $\phi_1(\mathbf{x},t)$ and $\mathbf{A}_1(\mathbf{x},t)$ only. Variation of $\mathcal{A}_R$ with respect to $\delta\mathcal{F}(\mathcal{Z})$, $\delta\phi_1(\mathbf{x},t)$, and $\delta\mathbf{A}_1(\mathbf{x},t)$ yields

$$\delta\mathcal{A}_R = -\int d^8\mathcal{Z}\Bigg[\delta\mathcal{F}(\mathcal{Z})\mathcal{H} + \mathcal{F}(\mathcal{Z}) \times\int d^3x\left(\delta\phi_1(\mathbf{x})\frac{\delta H}{\delta\phi_1(\mathbf{x})} + \delta\mathbf{A}_1(\mathbf{x})\cdot\frac{\partial H}{\partial\mathbf{A}_1(\mathbf{x})}\right)\Bigg] + \int\frac{d^4x}{4\pi}(\epsilon\nabla\,\delta\phi_1\cdot\nabla\Phi - \epsilon\nabla\times\delta\mathbf{A}_1\cdot\mathbf{B}). \tag{31}$$

Here, the variation $\delta\mathcal{F}$ is constrained to be of the form

$$\delta\mathcal{F} \equiv \{\mathcal{S},\mathcal{F}\}_{\mathcal{Z}}, \tag{32}$$

where $\mathcal{S}$ generates the virtual extended phase-space displacement $\delta\mathcal{Z} \equiv \{\mathcal{Z},\mathcal{S}\}_{\mathcal{Z}}$ and $\{\ ,\ \}_{\mathcal{Z}}$ is the Poisson bracket (18) on extended gyrocenter phase space. The functional derivatives $\delta H/\delta\phi_1(\mathbf{x})$ and $\delta H/\delta\mathbf{A}_1(\mathbf{x})$ in (31), on the other hand, are evaluated using (26) (to second order in ϵ) as

$$\frac{\partial H}{\partial\phi_1(\mathbf{x})} = \epsilon e\left\langle\left(\delta^3_{gc} - \epsilon\frac{e}{\Omega_0}\{\tilde{\Psi}_{1gc},\delta^3_{gc}\}\right)\right\rangle \equiv \epsilon e\langle(T^*_{\rm gy})^{-1}\delta^3_{gc}\rangle, \tag{33}$$

where $\delta^3_{gc} \equiv \delta^3(\mathbf{x}-\mathbf{R}-\boldsymbol{\rho}) \equiv \delta\phi_1(\mathbf{R}+\boldsymbol{\rho})/\delta\phi_1(\mathbf{x})$ using the identity

$$\phi_1(\mathbf{R}+\boldsymbol{\rho}) = \int d^3x\,\delta^3(\mathbf{x}-\mathbf{R}-\boldsymbol{\rho})\,\phi_1(\mathbf{x}) \equiv \delta\phi_{1gc}(\mathbf{R};\mu,\theta).$$

Similarly, we find

$$\frac{\partial H}{\partial\mathbf{A}_1(\mathbf{x})} = -\epsilon e\left\langle\left[\frac{\mathbf{v}}{c}\delta^3_{gc} - \epsilon\left(\frac{e}{\Omega_0}\left\{\tilde{\Psi}_{1gc},\frac{\mathbf{v}}{c}\delta^3_{gc}\right\} + \frac{e}{c}\mathbf{A}_{1gc}\cdot\left\{\mathbf{R}+\boldsymbol{\rho},\frac{\mathbf{v}}{c}\delta^3_{gc}\right\}\right)\right]\right\rangle \equiv -\epsilon e\left\langle(T^*_{\rm gy})^{-1}\left(\frac{\mathbf{v}}{c}\delta^3_{gc}\right)\right\rangle. \tag{34}$$

After rearranging and integrating by parts, (31) becomes

$$\delta\mathcal{A}_R = -\int d^4x\,\epsilon\frac{\delta\phi_1(\mathbf{x})}{4\pi}\left[\nabla^2\Phi + 4\pi e\times\int d^6Z f(Z,t)\langle(T^*_{\rm gy})^{-1}\delta^3_{gc}\rangle\right] + \int d^4x\,\epsilon\frac{\delta\mathbf{A}_1(\mathbf{x})}{4\pi}\cdot\left[\nabla\times\mathbf{B} - 4\pi e\int d^6Z f(Z,t)\left\langle(T^*_{\rm gy})^{-1}\left(\frac{\mathbf{v}}{c}\delta^3_{gc}\right)\right\rangle\right] + \int d^8\mathcal{Z}\mathcal{S}\{\mathcal{F},(w-H)\}_{\mathcal{Z}} + \int d^4x(\partial\cdot\mathcal{J}), \tag{35}$$

where we have used $\mathcal{F}(\mathcal{Z}) \equiv \delta(w-H)f(Z,t)$ for the gyrocenter Vlasov distribution function in extended phase space in the first two terms in $\delta\mathcal{A}$, while the last term in (35) involves the exact space-time divergence

$$\partial\cdot\mathcal{J}(x) \equiv \frac{\partial}{\partial x^\mu}\left(\int d^8\mathcal{Z}\delta^4(x-R)\mathcal{S}\mathcal{F}\dot{R}^\mu\right) + \nabla\cdot\left(\epsilon\frac{\delta\phi_1}{4\pi}\nabla\Phi - \epsilon\frac{\delta\mathbf{A}_1}{4\pi}\times\mathbf{B}\right), \tag{36}$$

where $\dot{R}^\mu \equiv \{R^\mu,\mathcal{H}\}_{\mathcal{Z}}$ denotes the gyrocenter four-velocity. Since (36) is an exact space–time divergence, it does not contribute to the reduced variational principle $\delta\mathcal{A}_R \equiv 0$.

By requiring that the action functional $\mathcal{A}_R$ be stationary with respect to arbitrary variations $\mathcal{S}$, $\delta\phi_1$, and $\delta\mathbf{A}_1$ (which vanish on the integration boundaries), we find the nonlinear gyrokinetic Vlasov equation

$$0 = \{\mathcal{F},\mathcal{H}\}_{\mathcal{Z}}, \tag{37}$$

and the gyrokinetic Maxwell equations; the gyrokinetic Poisson equation

$$\nabla^2\Phi(\mathbf{x}) = -4\pi\int d^6Z f(Z)\langle(T^*_{\rm gy})^{-1}e\,\delta^3(\mathbf{x}-\mathbf{R}-\boldsymbol{\rho})\rangle, \tag{38}$$

and the gyrokinetic Ampere equation,

$$\nabla\times\mathbf{B}(\mathbf{x}) = \frac{4\pi}{c}\int d^6Z f(Z)\langle(T^*_{\rm gy})^{-1}(e\mathbf{v}\delta^3(\mathbf{x}-\mathbf{R}-\boldsymbol{\rho}))\rangle. \tag{39}$$

If we now substitute $\mathcal{F}(\mathcal{Z}) \equiv \delta(w-H)f(Z,t)$ into $\{\mathcal{F},\mathcal{H}\}_{\mathcal{Z}} = 0$, we obtain the standard nonlinear gyrokinetic Vlasov equation $\partial_t f + \{f,H\} = 0$, written explicitly as

$$\frac{\partial f}{\partial t} + \left(\frac{\mathbf{B}^*}{B^*_\|}\frac{\partial H}{\partial p_\|} + \frac{c\hat{\mathbf{b}}}{eB^*_\|}\times\nabla H\right)\cdot\nabla f - \left(\frac{\mathbf{B}^*}{B^*_\|}\cdot\nabla H\right)\frac{\partial f}{\partial p_\|} = 0. \tag{40}$$

The nonlinear equations (38), (39), and (40) [with H given by (26)] are the self-consistent nonlinear gyrokinetic Vlasov–Maxwell equations in general magnetic field geometry.[4]

IV. LOCAL GYROKINETIC ENERGY CONSERVATION LAW BY THE NOETHER METHOD

By substituting (37), (38), and (39) into (35), the variational equation $\delta\mathcal{A} \equiv \int\delta\mathcal{L}d^4x$ (we henceforth ignore the subscript R) yields the Noether equation

$$\delta\mathcal{L}(x) \equiv \partial\cdot\mathcal{J}(x). \tag{41}$$

In the Noether method, the variations $(\mathcal{S},\delta\phi_1,\delta\mathbf{A}_1,\delta\mathcal{L})$ are expressed in terms of generators for infinitesimal translations in space or time.

Following a translation in time $t \rightarrow t+\delta t$, the variations $\mathcal{S}$, $\delta\phi_1$, $\delta\mathbf{A}_1$, and $\delta\mathcal{L}$ become, respectively,

$$\begin{aligned}
&\mathcal{S}=-w\,\delta t,\\
&\delta\phi_1=-\delta t\,\partial_t\phi_1,\\
&\delta\mathbf{A}_1=-\delta t\,\partial_t\mathbf{A}_1\equiv c\,\delta t(\mathbf{E}+\nabla\Phi),\\
&\delta\mathcal{L}=-\delta t\,\partial_t\mathcal{L}.
\end{aligned}\tag{42}$$

In (42), the expression for $\mathcal{S}$ satisfies $\delta t\equiv\{t,\mathcal{S}\}_{\mathcal{Z}}$, the expression for $\delta\mathbf{A}_1$ incorporates the identity $\mathbf{E}_0+\nabla\phi_0\equiv 0$, and the Vlasov–Maxwell Lagrangian density is

$$\mathcal{L}=\frac{1}{8\pi}(|\nabla\Phi|^2-|\mathbf{B}|^2)\equiv\frac{1}{4\pi}|\nabla\Phi|^2-\mathcal{E}_{EM},$$

after the physical constraint $\mathcal{H}=0$ is imposed in the space–time integrand of the reduced action functional (30); here, $\mathcal{E}_{EM}\equiv(|\nabla\Phi|^2+|\mathbf{B}|^2)/8\pi$ denotes the electromagnetic-field energy density (in the low-frequency limit).

By combining (42) into (36) and (41), we obtain

$$\begin{aligned}
0=&\frac{\partial}{\partial t}\left(\frac{1}{4\pi}\overbrace{|\nabla\Phi|^2}^{\text{(I)}}-\mathcal{E}_{EM}-\int d^6Z\,\delta^3(\mathbf{x}-\mathbf{R})Hf\right)\\
&-\nabla\cdot\left[\int d^6Z\,\delta^3(\mathbf{x}-\mathbf{R})Hf\dot{\mathbf{R}}\right]\\
&+\nabla\cdot\left[-\frac{\epsilon}{4\pi}\overbrace{\partial_t\phi_1\nabla\Phi}^{\text{(II)}}-\frac{c}{4\pi}(\mathbf{E}+\overbrace{\nabla\Phi}^{\text{(III)}})\times\mathbf{B}\right].
\end{aligned}\tag{43}$$

The term (I) can be written as

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{1}{4\pi}|\Phi|^2\right)=&\frac{\partial}{\partial t}\left[\nabla\cdot\left(\frac{1}{4\pi}\Phi\nabla\Phi\right)\right.\\
&\left.+e\int d^6Z\,\delta^3(\mathbf{x}-\mathbf{R})f\langle(T^*_{\rm gy})^{-1}\Phi_{gc}\rangle\right],
\end{aligned}\tag{44}$$

where $\Phi_{gc}=\phi_0+\epsilon\phi_{1gc}$, the identity $|\nabla\Phi|^2\equiv\nabla\cdot(\Phi\nabla\Phi)-\Phi\nabla^2\Phi$ was used, and (38) was substituted. The term (II) can be written as

$$\begin{aligned}
-\nabla\cdot\left(\frac{\epsilon}{4\pi}\partial_t\phi_1\nabla\Phi\right)=&-\nabla\frac{\partial}{\partial t}\cdot\left(\frac{\epsilon}{4\pi}\phi_1\nabla\Phi\right)\\
&+\nabla\cdot\left(\frac{\epsilon^2}{4\pi}\phi_1\nabla\partial_t\phi_1\right).
\end{aligned}\tag{45}$$

Lastly, the term (III) can be written as

$$\begin{aligned}
-\nabla\cdot\left(\nabla\Phi\times\frac{c}{4\pi}\mathbf{B}\right)&=\nabla\cdot\left(\Phi\frac{c}{4\pi}\nabla\times\mathbf{B}\right)\\
&=\nabla\cdot\left[e\int d^6Zf\langle(T^*_{\rm gy})^{-1}(\mathbf{v}\Phi_{gc})\rangle\right],
\end{aligned}\tag{46}$$

where (39) was substituted. We now note that the first term in (45) partially cancels the first term in (44) leaving

$$\frac{\partial}{\partial t}\nabla\cdot\left(\frac{\phi_0}{4\pi}\nabla\Phi\right)\equiv\nabla\cdot\left(\epsilon\frac{\phi_0}{4\pi}\nabla\partial_t\phi_1\right).$$

By collecting the remaining terms, we obtain the following expression for the *local* gyrokinetic energy conservation law,

$$\frac{\partial\mathcal{E}}{\partial t}+\nabla\cdot\mathbf{S}=0,\tag{47}$$

where the gyrokinetic energy density is

$$\begin{aligned}
\mathcal{E}(\mathbf{x},t)=&\int d^6Z\,\delta^3(\mathbf{x}-\mathbf{R})f(Z,t)(H-e\langle(T^*_{\rm gy})^{-1}\Phi_{gc}\rangle)\\
&+\frac{1}{8\pi}(|\nabla\Phi|^2+|\mathbf{B}|^2),
\end{aligned}\tag{48}$$

while the gyrokinetic energy density flux is

$$\begin{aligned}
\mathbf{S}(\mathbf{x},t)=&\int d^6Z\,\delta^3(\mathbf{x}-\mathbf{R})f(Z,t)(H\dot{\mathbf{R}}-e\langle(T^*_{\rm gy})^{-1}\mathbf{v}\Phi_{gc}\rangle)\\
&+\frac{1}{4\pi}\left(c\mathbf{E}\times\mathbf{B}-\epsilon\frac{\Phi}{4\pi}\nabla\partial_t\phi_1\right).
\end{aligned}\tag{49}$$

The last term in (49) represents the contribution from the polarization current. Integrating (47) over space we recover the standard *global* gyrokinetic energy conservation law[4,15] $dE/dt=0$, where

$$\begin{aligned}
E\equiv\int\mathcal{E}(\mathbf{x},t)d^3x=&\int\frac{d^3x}{8\pi}(|\nabla\Phi|^2+|\mathbf{B}|^2)+\int d^6Zf(Z,t)\\
&\times(H-e\langle(T^*_{\rm gy})^{-1}\Phi_{gc}\rangle)
\end{aligned}\tag{50}$$

is the total gyrokinetic Vlasov–Maxwell energy.

V. SUMMARY AND FUTURE WORK

In the present work, the Eulerian variational principle for low-frequency nonlinear gyrokinetic Vlasov–Maxwell equations is presented for the first time. This new variational principle is based on constrained variations of the Vlasov distribution function $\mathcal{F}$ on extended eight-dimensional phase space. The variational principle is also an extension of the Eulerian variational principle previously constructed for the linearized gyrokinetic Vlasov–Maxwell equations.[14] It also provides a simpler variational principle than the Lagrangian variational principle presented by Sugama[13] in which the gyrocenter phase-space coordinates are variational variables.

The elegance of the present Eulerian variational principle is demonstrated by the ease with which self-consistent reduced Maxwell equations can be derived, e.g., through the functional derivative $\delta\mathcal{A}_{VR}/\delta A_\nu(x)$ in (9). As a further application, we also derive (for the first time) a local energy conservation law for the nonlinear gyrokinetic Vlasov–Maxwell equations from which the standard global gyrokinetic energy invariant (50) is recovered.

In future work, the reduced Eulerian variational formalism will be applied to derive self-consistent, nonlinear, low-frequency bounce-gyrokinetic Vlasov–Maxwell equations, in which both the fast gyromotion and bounce-motion time scales have been eliminated. This work will be based on the recent derivation of nonlinear bounce-gyrocenter Hamiltonian dynamics.[17]

ACKNOWLEDGMENTS

The author acknowledges useful discussions with Professor Allan Kaufman.

This work was supported by the United States Department of Energy under Contract No. DE-AC03-76SFOO098.

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