Nonlinear bounce-gyrocenter Hamiltonian dynamics in general magnetic field geometry

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Nonlinear bounce-gyrocenter Hamilton equations for full electromagnetic field fluctuations in general magnetic geometry are derived by the phase-space Lagrangian Lie-perturbation method. These reduced dynamical equations can be used to follow the orbits of magnetically trapped charged particles in the presence of low-frequency electromagnetic fluctuations in magnetic-field geometries suitable for applications in fusion and space plasma physics. © *2000 American Institute of Physics.* [S1070-664X(00)01308-2]

I. INTRODUCTION

The magnetic confinement of charged particles implies the existence of orbits enclosed within a compact volume in space, which in turn generically allows the existence of three orbital frequencies.[1] The first of these orbital frequencies, called the gyrofrequency (denoted ω_c), exists even in uniform (unconfining) magnetic fields and describes the gyration of a charged particle about a single magnetic field line. The second orbital frequency, called the bounce frequency (denoted ω_b), requires longitudinal confinement along magnetic field lines (due to nonuniformity parallel to the field lines) and describes oscillations in the parallel component of the particle's velocity, which vanishes at turning points along the *trapped*-particle orbit. Although certain magnetic geometries (e.g., axisymmetric tokamak geometry)[2] allow for the existence of confined, untrapped (or circulating) charged particles whose parallel velocity exhibit oscillatory behavior about a nonvanishing value, we focus our attention only on trapped-particle orbits in the present work. The third orbital frequency, called the drift-precession frequency (denoted ω_d), describes the periodic drift motion across magnetic field lines (e.g., due to magnetic curvature). In general, these three orbital frequencies are widely separated (for a single particle species), with $\omega_c \gg \omega_b \gg \omega_d$.

When the characteristic time scale of interest (denoted τ) is much longer than the gyroperiod (i.e., when the particle has executed many gyration cycles during time τ), the fast gyration angle can be asymptotically removed from the particle's orbital dynamics and a corresponding adiabatic invariant (the magnetic moment μ) can be constructed.[1] The resulting *guiding-center* dynamics takes place in a reduced four-dimensional phase space with noncanonical coordinates $(\mathbf{X},p_\parallel)$, where $\mathbf{X}$ denotes the particle's guiding-center position and $p_\parallel$ denotes its parallel kinetic momentum. Guiding-center dynamics has been shown to possess a noncanonical Hamiltonian structure,[3,4] i.e., $\dot{\mathbf{X}} \equiv \{\mathbf{X}, H_{gc}\}_{gc}$ and $\dot{p}_\parallel \equiv \{p_\parallel, H_{gc}\}_{gc}$ are expressed in terms of a guiding-center Hamiltonian function H_{gc} and a noncanonical guiding-center Poisson bracket $\{\ \}_{gc}$; here and throughout the paper, a dot denotes a time derivative.

When the characteristic time scale τ is also much longer than the bounce period (i.e., when the particle has executed many bounce cycles during time τ), the fast bounce angle can be asymptotically removed from the particle's orbital dynamics and a corresponding adiabatic invariant (the longitudinal or bounce action J) can be constructed. The resulting *bounce-averaged guiding-center* dynamics takes place in a reduced two-dimensional phase space with spatial coordinates (y^1,y^2), where each coordinate y^a (with $a=1$ or 2) satisfies the condition $\mathbf{B}\cdot\nabla y^a=0$; the coordinates (y^1,y^2) are known as magnetic field line *labels*. Bounce-averaged guiding-center dynamics in static magnetic fields has also been shown to possess a canonical Hamiltonian structure[5] (see Sec. III).

In the present paper we are interested in constructing nonlinear Hamilton equations for charged particles in the presence of low-frequency electromagnetic fluctuations with characteristic mode frequency $\omega \equiv 2\pi/\tau$ such that

$$\omega_d,\ \omega \ll \omega_b \ll \omega_c, \tag{1}$$

i.e., the characteristic time scale is much longer than the gyration and bounce orbital time scales. This time-scale ordering thus allows the removal of the fast gyration and bounce angles, i.e., the reduced dynamics preserves the invariance of the magnetic moment μ and the bounce action J.

In deriving these reduced equations, we introduce two different orderings represented by the small dimensionless parameters ϵ_d and ϵ_ω. The first small parameter (ϵ_d) represents the ratio of the fast bounce-motion time scale to the slow drift-motion time scale in a nonuniform background magnetic field (i.e., $\omega_d \ll \omega_b$). The second small parameter (ϵ_ω) represents the ratio of the fast bounce-time scale to the slow characteristic time scale associated with electromagnetic field perturbations (i.e., $\omega \ll \omega_b$). Such perturbation fields, whose amplitudes are represented by the dimensionless ordering parameter ϵ_δ, have characteristic frequency ω and characteristic parallel and perpendicular wave numbers $k_\parallel$ and $k_\perp$, respectively, with respect to the background magnetic field. The reduced Hamilton equations derived here contain terms up to order ϵ_δ^2.

In the present paper, we ignore finite-Larmor-radius effects associated with the electromagnetic field perturbations (i.e., we take the limit of small gyroradius, $k_\perp^2 \rho^2 \ll 1$), and we refrain from ordering the perpendicular and parallel wave numbers, since $k_\parallel / k_\perp$ may not be small. Next, we ignore relativistic effects even if these effects might be relevant for certain class of trapped particles; nonlinear relativistic gyrokinetic equations have recently been derived,[6] and future work will undoubtedly focus on the derivation of nonlinear relativistic bounce dynamics. We also ignore any background static electric fields, i.e., any electric field is automatically viewed as the result of electromagnetic field fluctuations.

We adopt a two-step transformation scheme. First, the bounce-angle dependence due to the background (static) magnetic field is asymptotically eliminated with the help of the bounce-averaged guiding-center (bgc) transformation; in the process, a bounce-action adiabatic invariant is constructed as an asymptotic expansion in powers of ϵ_d. Next, low-frequency electromagnetic field fluctuations are introduced into the unperturbed bgc phase-space Lagrangian. These perturbations reintroduce bounce-angle dependence in the bgc phase-space Lagrangian and thus the bgc bounce action is no longer invariant. To construct a new bounce-action invariant, we proceed with the bounce-averaged gyrocenter (bgy) phase-space transformation which asymptotically eliminates the bounce-angle dependence introduced by the field perturbations. In this process, a new bgy bounce-action adiabatic invariant is constructed as an asymptotic expansion in powers of ϵ_δ.

The remainder of this paper is organized as follows. In Sec. II, we present the mathematical foundations for the construction of general magnetic geometries. The notation introduced in this section will be used throughout the paper. In Sec. III, the Hamiltonian theory of bounce-guiding-center motion in static magnetic geometry is presented; only results are presented and the reader is refered to the original work of Littlejohn[5] for further details. Magnetic coordinates introduced in Sec. II are used here to describe the unperturbed bounce and drift motions.

In Sec. IV, electromagnetic field perturbations are introduced and shown to destroy the adiabatic invariance of the unperturbed bounce action by re-introducing bounce-angle dependence in the perturbed bounce guiding-center Hamiltonian system. Here, the electromagnetic field fluctuations are represented by the perturbed scalar potential $\delta\phi$, the parallel component of the perturbed vector potential $\delta A_\parallel$, and the parallel component of the perturbed magnetic field $\delta B_\parallel$. The phase-space Lagrangian Lie-perturbation method[7] is used to remove this new bounce-angle dependence and construct new bounce-gyrocenter coordinates. In particular, the new bounce-gyrocenter action is preserved by the perturbed bounce-averaged Hamiltonian dynamics. Section V summarizes our work and discusses possible applications in fusion and space plasma physics. Last, the Appendix presents two important magnetic geometries: the axisymmetric tokamak geometry (for fusion plasma applications) and the distorted dipole geometry (for space plasma applications).

II. MAGNETIC FIELD GEOMETRY

This section focuses on the general representation of magnetic fields in terms of Euler and Clebsch potentials and introduces the definition of magnetic geometries in terms of magnetic coordinates. After having read this section, the reader may consult the Appendix for explicit examples of magnetic geometries suitable for applications in fusion and space plasma physics.

A. Magnetic field representations

In general, a magnetic field is divergenceless and can be written (at least locally) as

$$\mathbf{B} \equiv \nabla \xi \times \nabla \psi, \tag{2}$$

where ξ and ψ are called Euler potentials.[8] According to (2), each magnetic field line is labeled by ψ and ξ (since $\mathbf{B}\cdot\nabla\psi = 0 = \mathbf{B}\cdot\nabla\xi$), and the magnetic vector potential $\mathbf{A}$ (with $\mathbf{B} \equiv \nabla\times\mathbf{A}$) can be written as

$$\mathbf{A} \equiv \tfrac{1}{2}(\xi\,\nabla\psi - \psi\,\nabla\xi) + \nabla\alpha, \tag{3}$$

where the gauge function α (which may be multivalued) is involved in the definition of magnetic helicity $\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\nabla\alpha$. Since the gauge term $\nabla\alpha$ does not play a role in what follows [see the discussion following (12)], however, we henceforth omit it.

Another useful representation for the magnetic field is the covariant (Clebsch) representation

$$\mathbf{B} \equiv \sum_i \lambda_i \,\nabla\chi^i, \tag{4}$$

where (λ_i, χ^i) are Clebsch potentials[9] (here the index i goes from 1 to 3); note that these potentials must still satisfy the condition $\nabla\cdot\mathbf{B} = 0$. The Clebsch representation is useful when an explicit expression for $\nabla\times\mathbf{B} \equiv \Sigma_i \nabla\lambda_i \times \nabla\chi^i$ is known. For example, if $\nabla\times\mathbf{B} \equiv 0$, then the magnetic field $\mathbf{B}$ can simply be written as $\mathbf{B} \equiv \nabla\chi$.

B. Magnetic coordinates and magnetic geometry

1. Magnetic coordinates

The magnetic field representations (2) and (4) allow the introduction of the magnetic *coordinates* $\Psi^i \equiv (\xi, \psi, s)$, where ξ and ψ are the Euler potentials for $\mathbf{B}$ and s is the spatial coordinate along a magnetic field line:

$$\frac{\partial \mathbf{X}}{\partial s} \equiv \hat{\mathbf{b}} \equiv \frac{\mathbf{B}}{B}. \tag{5}$$

Using the notation $\mathbf{y} \equiv (\xi, \psi)$ for coordinates in the space of field-line labels (i.e., each magnetic field line is represented as a point in $\mathbf{y}$ space), the magnetic field (2) and the magnetic vector potential (3) can also be written as

$$\mathbf{B} \equiv \tfrac{1}{2}\eta_{ab}\,\nabla y^a \times \nabla y^b, \quad \mathbf{A} \equiv \tfrac{1}{2}\eta_{ab}\, y^a\, \nabla y^b, \tag{6}$$

where η_{ab} is antisymmetric in its indices (with $\eta_{12} = +1 = -\eta_{21}$).

To define a magnetic geometry, we require the complete sets of contravariant basis vectors $(\partial\mathbf{X}/\partial\Psi^i)$ and covariant basis vectors $(\nabla\Psi^i)$. Since the vectors ∇y^a are given in (6) and $\partial\mathbf{X}/\partial s$ is given by (5), we only need expressions for ∇s and $\partial\mathbf{X}/\partial y^a$. In deriving these expressions, we use the orthogonality relations

$$\nabla\Psi^i\cdot\frac{\partial\mathbf{X}}{\partial\Psi^j}=\delta^i_j, \tag{7}$$

between the contravariant and covariant basis vectors. Using these relations, we obtain the following expression for ∇s:

$$\nabla s\equiv\hat{\mathbf{b}}-\sum_a \mathcal{R}_a\nabla y^a, \tag{8a}$$

where

$$\mathcal{R}_a\equiv\hat{\mathbf{b}}\cdot\frac{\partial\mathbf{X}}{\partial y^a}=\sum_i \frac{\lambda_i}{B}\frac{\partial\chi^i}{\partial y^a}, \tag{8b}$$

while we find for $\partial\mathbf{X}/\partial y^a$:

$$\frac{\partial\mathbf{X}}{\partial y^a}\equiv\mathcal{R}_a\,\hat{\mathbf{b}}+\sum_b \eta_{ab}\,\nabla y^b\times\frac{\mathbf{B}}{B^2}. \tag{9}$$

It is now quite simple to check that the sets $(\nabla y^a,\nabla s)$ and $(\partial\mathbf{X}/\partial y^a,\partial\mathbf{X}/\partial s)$ satisfy the orthogonality relations (7).

Next, we construct the parallel gradient operator $\partial_\parallel\equiv\hat{\mathbf{b}}\cdot\nabla=\partial/\partial s$ and the perpendicular gradient operator $\nabla_\perp\equiv-\hat{\mathbf{b}}\times(\hat{\mathbf{b}}\times\nabla)$, which only has y^a components:

$$\frac{\partial\mathbf{X}}{\partial y^a}\cdot\nabla_\perp\equiv\frac{\partial}{\partial y^a}-\mathcal{R}_a\frac{\partial}{\partial s}\equiv\partial_{\perp a}.$$

Hence the gradient operator can be expressed as $\nabla\equiv\hat{\mathbf{b}}\,\partial_\parallel+\nabla y^a\,\partial_{\perp a}$. Last, the Jacobian for the transformation $\mathbf{X}\rightarrow(\xi,\psi,s)$ is

$$\frac{\partial\mathbf{X}}{\partial\xi}\times\frac{\partial\mathbf{X}}{\partial\psi}\cdot\frac{\partial\mathbf{X}}{\partial s}\equiv B^{-1}\equiv(\nabla\xi\times\nabla\psi\cdot\nabla s)^{-1}, \tag{10}$$

so that $d^3X\equiv B^{-1}\,d^2y\,ds$ is the infinitesimal volume element in magnetic coordinates.

2. Magnetic geometry

A magnetic *geometry* is defined in terms of the magnetic coordinates (ψ,ξ,s) and the contravariant and covariant basis vectors $(\partial\mathbf{X}/\partial\Psi^i)$ and $(\nabla\Psi^i)$, respectively. The covariant-basis vectors yield the following components $g^{ij}\equiv\nabla\Psi^i\cdot\nabla\Psi^j$ for the two-contravariant metric tensor:

$$\begin{aligned} g^{ab}&\equiv\nabla y^a\cdot\nabla y^b,\\ g^{as}&\equiv\nabla y^a\cdot\nabla s=-g^{ab}\mathcal{R}_b,\\ g^{ss}&\equiv|\nabla s|^2=1+\mathcal{R}_a g^{ab}\mathcal{R}_b, \end{aligned} \tag{11a}$$

where summation over repeated indices is henceforth assumed, the identity $\hat{\mathbf{b}}\cdot\nabla y^a\equiv 0$ was used, and (8a) was used for ∇s. The contravariant-basis vectors yield the following components $g_{ij}\equiv\partial\mathbf{X}/\partial\Psi^i\cdot\partial\mathbf{X}/\partial\Psi^j$ for the two-covariant metric tensor:

$$\begin{aligned} g_{ab}&\equiv\partial\mathbf{X}/\partial y^a\cdot\partial\mathbf{X}/\partial y^b=\mathcal{R}_a\mathcal{R}_b-B^{-2}\,\eta_{ac}g^{cd}\,\eta_{db},\\ g_{as}&\equiv\partial\mathbf{X}/\partial y^a\cdot\partial\mathbf{X}/\partial s=\mathcal{R}_a,\\ g_{ss}&\equiv|\partial\mathbf{X}/\partial s|^2=1. \end{aligned} \tag{11b}$$

The components $g^j_i\equiv\partial\mathbf{X}/\partial\Psi^i\cdot\nabla\Psi^j=\delta^j_i$ of the mixed metric tensor are given by the orthogonality relations (7).

Last, by definition, we have

$$\nabla\Psi^i\equiv g^{ij}\frac{\partial\mathbf{X}}{\partial\Psi^j},\quad \frac{\partial\mathbf{X}}{\partial\Psi^i}\equiv g_{ij}\nabla\Psi^j,$$

while $\nabla\cdot\mathbf{A}\equiv B\partial_i(B^{-1}A^i)$, where $A^i\equiv\mathbf{A}\cdot\nabla\Psi^i$, and $(\nabla\times\mathbf{A})^i\equiv\epsilon^{ijk}B\partial_jA_k$, where $A_i\equiv\mathbf{A}\cdot\partial\mathbf{X}/\partial\Psi^i$. We note that the determinant of the two-contravariant metric tensor g^{ij} is equal to B^2, the determinant of the two-covariant metric tensor g_{ij} is equal to B^{-2}, and that $g^{ij}g_{jk}\equiv\delta^i_k$. The metric tensors defined here can then be used to define the magnetic geometry (e.g., we can construct Christoffel symbols, etc.).

III. UNPERTURBED BOUNCE-GUIDING-CENTER MOTION

In this section, we consider the motion of charged particles in a static nonuniform magnetic field $\mathbf{B}_0\equiv B_0\hat{\mathbf{b}}_0$ (the subscript 0 is used here to distinguish the unperturbed background magnetic field from the fluctuating magnetic field $\delta\mathbf{B}$). To lowest order in ρ/L_B (where L_B is the background magnetic-field length scale), the unperturbed guiding-center phase-space Lagrangian for guiding-center particles (of mass m and charge q) is[4]

$$\gamma_0=\frac{q}{c}\mathbf{A}_0\cdot d\mathbf{X}+p_\parallel\hat{\mathbf{b}}_0\cdot d\mathbf{X}-\left(\mu B_0+\frac{p_\parallel^2}{2m}\right)dt, \tag{12}$$

where $(\mathbf{X},p_\parallel;\mu)$ are the guiding-center reduced phase-space coordinates: $\mathbf{X}\equiv(\mathbf{y},s)$ gives the guiding-center position in magnetic-coordinate space, $p_\parallel=mv_\parallel$ is the parallel guiding-center momentum, and μ is the guiding-center magnetic moment (an exact dynamical invariant in guiding-center Hamiltonian theory). From (3) and (7), we find

$$\mathbf{A}_0\cdot d\mathbf{X}\equiv(1/2)\,\eta_{ab}y^a\,dy^b+d\alpha,$$

where the last term comes from the gauge term $\nabla\alpha$ in $\mathbf{A}_0$. Since the phase-space Lagrangian (12) is only defined up to an exact derivative, however, we can omit the gauge term $d\alpha$ in $\mathbf{A}_0\cdot d\mathbf{X}$. Using (8a), we also find $\hat{\mathbf{b}}_0\cdot d\mathbf{X}\equiv ds+\mathcal{R}_a\,dy^a$. The unperturbed guiding-center phase-space Lagrangian (12) can therefore be written as

$$\begin{aligned} \gamma_0&=\left(\frac{q}{2c\,\epsilon_d}\,\eta_{ab}y^a+p_\parallel\mathcal{R}_b\right)dy^b+p_\parallel\,ds\\ &\quad-\left(\mu B_0(\mathbf{y},s)+\frac{p_\parallel^2}{2m}\right)dt\\ &\equiv\mathcal{F}_b\,dy^b+p_\parallel\,ds-H_0\,dt, \end{aligned} \tag{13}$$

where $\epsilon_d\ll 1$ is introduced as an ordering parameter representing the ratio of the fast bounce time scale to the slow drift time scale [see the discussion following (15)].

From the phase-space Lagrangian (13), expressed as $\gamma_0\equiv L_0\,dt$, the Euler–Lagrange equations associated with the guiding-center Lagrangian $L_0\equiv\mathcal{F}_b\dot{y}^b+p_\parallel\dot{s}-H_0$ lead to the following dynamical equations: the components for the guiding-center drift velocity are

$$\dot{y}^a = -\epsilon_d\left(\frac{c/q}{1+\epsilon_d\Delta}\right)\eta^{ab}(\mu\partial_{\perp b}B_0 + mv_\parallel^2\partial_\parallel\mathcal{R}_b), \tag{14a}$$

where $\eta^{ab} \equiv \eta_{ab}^{-1} = -\eta_{ab}$ and $\Delta \equiv (cp_\parallel/qB_0)\hat{\mathbf{b}}_0\cdot\nabla\times\hat{\mathbf{b}}_0 = (cp_\parallel/q)\eta^{ab}\partial_{\perp b}\mathcal{R}_a$; the guiding-center parallel velocity equation is

$$v_\parallel = \dot{s} + \mathcal{R}_a\dot{y}^a, \tag{14b}$$

and the acceleration of the guiding-center parallel velocity is

$$\dot{v}_\parallel = -(\mu/m)\partial_\parallel B_0 + (v_\parallel\partial_\parallel\mathcal{R}_a)\dot{y}^a. \tag{14c}$$

We note that the dynamical equations (14a)–(14c) conserve the particle energy $E_0 \equiv p_\parallel^2/2m + \mu B_0$ exactly, i.e.,

$$\frac{dE_0}{dt} = \dot{y}^a(\mu\partial_{\perp a}B_0 + mv_\parallel^2\partial_\parallel\mathcal{R}_a) \equiv 0, \tag{14d}$$

where we used (14a) and the antisymmetry of η^{ab}. Note also that $v_\parallel \neq \dot{s}$ since the spatial coordinates $(\mathbf{y},s)$ are nonorthogonal (i.e., $g_{ij} \neq 0$ for $i \neq j$).

A. Preliminary phase-space tranformation

To lowest order in the ϵ_d ordering, we see that the drift velocity (14a) vanishes while (14b) and (14c) yield

$$\dot{s} = v_\parallel, \quad \dot{v}_\parallel = -(\mu/m)\partial_\parallel B_0. \tag{15}$$

Hence, to lowest order in ϵ_d, the motion is taking place along a magnetic field line (labeled by $\mathbf{y}$) and drift motion is absent. Since the lowest-order bounce equations (15) are zeroth order in ϵ_d while the drift equations are first order, we see that the parameter ϵ_d is indeed the ratio of the fast bounce time scale over the slow drift time scale.

Bounce motion thus takes place in the $(s,v_\parallel)$ plane (henceforth known as the bounce plane), where

$$v_\parallel(s,E_0,\mu;\mathbf{y}) \equiv \pm\sqrt{\frac{2}{m}[E_0 - \mu B_0(s;\mathbf{y})]}.$$

Here, $\pm$ refers to the sign of $v_\parallel$ along the magnetic field, i.e., the particle moves either in the same direction $(+)$ or the opposite direction $(-)$ of the magnetic field. If $E_0 > \mu B_0(s)$ everywhere along the magnetic field line (i.e., for fixed $\mathbf{y}$), the parallel velocity $v_\parallel$ exhibits modulations in its amplitude and its sign is an invariant (i.e., the $\pm$ branches are separated); such a particle orbit is called *circulating*.

On the other hand, if at some point $s = s_0$ along a magnetic field line the condition $E_0 = \mu B_0(s_0)$ is satisfied, then the $\pm$ branches coincide at $v_\parallel = 0$ and the point s_0 is called a turning point. The particle bounces from this turning point (since $\dot{s}$ changes sign at that point) and moves back along the field line until it encounters another turning point, located say at $s = s_1$. Bounce motion is therefore periodic in both s and $v_\parallel$.

Following a standard procedure in classical mechanics,[10] one constructs action-angle canonical variables associated with this periodic motion. Here, the action-angle coordinates (J,ζ) associated with periodic bounce motion have the following lowest-order expressions: For the bounce action J, we find[1,5]

$$J(E_0,\mu;\mathbf{y}) \equiv \frac{1}{2\pi}\oint p_\parallel(s,E_0,\mu;\mathbf{y})ds$$
$$= \frac{1}{\pi}\int_{s_0}^{s_1}\sqrt{2m[E_0 - \mu B_0(s;\mathbf{y})]}ds, \tag{16a}$$

where (s_0,s_1) are the turning points where $v_\parallel$ vanishes, while for the bounce angle ζ, we find[5]

$$\zeta(s,E_0,\mu;\mathbf{y}) \equiv \pi \pm \omega_b\int_{s_0}^{s}\frac{ds'}{\sqrt{\frac{2}{m}[E_0 - \mu B_0(s';\mathbf{y})]}}, \tag{16b}$$

where $\zeta(s=s_0) \equiv \pi$ for both branches. The bounce frequency ω_b is defined from (16a) as

$$\omega_b(\mathbf{y};E_0,\mu) \equiv \left(\frac{\partial J}{\partial E_0}\right)^{-1} = 2\pi\left(\oint\frac{ds}{v_\parallel}\right)^{-1}. \tag{16c}$$

From this expression, we see that the bounce frequency ω_b scales as $E_0^{1/2}$ in terms of the total energy E_0 and also includes a dependence on the ratio $\mu\bar{B}_0(\mathbf{y})/E_0$ (related to pitch angle), where $\bar{B}_0(\mathbf{y})$ is a (local) minimum value along the magnetic field line.

We can now proceed to perform the substitution $(s,p_\parallel) \to (J,\zeta)$ in the guiding-center phase-space Lagrangian (13). First, we note that the transformation $(s,p_\parallel) \to (J,\zeta) \equiv \mathbf{u}$ is canonical since

$$\frac{\partial p_\parallel}{\partial\zeta}\frac{\partial s}{\partial J} - \frac{\partial p_\parallel}{\partial J}\frac{\partial s}{\partial\zeta} = 1. \tag{17}$$

Next, we note that by definition $\partial s/\partial y^a \equiv 0$ and, hence, the parallel spatial coordinate s is a function of the bounce action-angle coordinates $\mathbf{u}$ only: $s = s(\mathbf{u})$. In (13), the differential ds becomes $ds = \partial_\alpha s\, du^\alpha$. The unperturbed guiding-center phase-space Lagrangian (13) becomes

$$\gamma_0 \equiv \left(\frac{q}{2c\epsilon_d}\eta_{ab}y^a + p_\parallel\mathcal{R}_b\right)dy^b$$
$$+ \left(p_\parallel\frac{\partial s}{\partial u^\alpha}\right)du^\alpha - H_0(\mathbf{y},\mathbf{u})dt, \tag{18}$$

where $H_0(\mathbf{y},\mathbf{u}) \equiv \mu B_0(\mathbf{y};s(\mathbf{u})) + [p_\parallel(\mathbf{y},\mathbf{u})]^2/2m$ is the lowest-order unperturbed guiding-center Hamiltonian and explicit bounce-angle dependence now appears in (18). The unperturbed guiding-center Hamiltonian H_0, however, is independent of the bounce angle ζ (to lowest order) since

$$\frac{\partial H_0}{\partial\zeta} \equiv \left(v_\parallel\frac{\partial p_\parallel}{\partial s} + \mu\frac{\partial B_0}{\partial s}\right)\frac{\partial s}{\partial\zeta} \equiv 0,$$

where we used the lowest-order expression (15) for the guiding-center parallel acceleration.

B. Bounce-guiding-center phase-space transformation

Because of its dependence on the field-line labels $\mathbf{y}$, the bounce action (16a) is not conserved at order ϵ_d [i.e., $dJ/dt = \mathcal{O}(\epsilon_d)$]. To remove the bounce-angle dependence in (18) and construct an asymptotic expansion for the bounce-action adiabatic invariant, we proceed by performing an infinitesimal phase-space transformation $(\mathbf{y},\mathbf{u})\rightarrow(\overline{\mathbf{y}},\overline{\mathbf{u}})$, where the relation between the guiding-center coordinates $(\mathbf{y},\mathbf{u})$ and the bounce guiding-center (bgc) coordinates $(\overline{\mathbf{y}},\overline{\mathbf{u}})$ is given in terms of the asymptotic expansions

$$y^a \equiv \overline{y}^a - \epsilon_d G_1^a + \cdots, \qquad u^\alpha \equiv \overline{u}^\alpha - \epsilon_d G_1^\alpha + \cdots, \tag{19}$$

where the components G_n^a and G_n^α of the nth-order generating vector field are constructed so that the bounce action $\overline{J} = J + \Sigma_{k=1}^n \epsilon_d^k G_k^J$ is conserved at the nth order, i.e., $d\overline{J}/dt = \mathcal{O}(\epsilon_d^{n+1})$. The $\mathbf{y}$ components of the first-order generating vector are[5]

$$G_1^a = -\eta^{ab}\frac{c}{q}\left(\frac{\partial S_1}{\partial \overline{y}^b} + p_\parallel \mathcal{R}_b\right), \tag{20a}$$

where the gauge function $S_1(\overline{\mathbf{y}},\overline{\mathbf{u}})$ is defined from the relation

$$\frac{\partial S_1}{\partial \overline{u}^\beta} \equiv -\frac{\eta_{\alpha\beta}}{2}\overline{u}^\alpha - p_\parallel(\overline{\mathbf{y}},\overline{\mathbf{u}})\frac{\partial s}{\partial \overline{u}^\beta}(\overline{\mathbf{u}}), \tag{20b}$$

with $\eta_{\alpha\beta}$ antisymmetric in its indices (with $\eta_{12} = +1$). The reader is referred to Ref. 5 for further details on the unperturbed bounce-guiding-center phase-space transformation; since the $\mathbf{u}$ components of the first-order generating vector (G_1^α) are not needed in what follows, we omit them here.

The purpose of this transformation is thus to remove the bounce-angle dependence at all orders in ϵ_d. Hence, the unperturbed bgc phase-space Lagrangian becomes

$$\overline{\gamma}_0 \equiv \frac{q}{2c\epsilon_d}\eta_{ab}\overline{y}^a d\overline{y}^b + \overline{J}\, d\overline{\zeta} - \overline{H}_0(\overline{\mathbf{y}},\overline{J};\epsilon_d)\,dt, \tag{21a}$$

and the unperturbed bgc Hamiltonian is[5]

$$\overline{H}_0(\overline{\mathbf{y}},\overline{J};\epsilon_d) \equiv H_0(\overline{\mathbf{y}},\overline{J}) - \frac{\epsilon_d}{2}\left(\omega_b \eta_{ab}\left\langle G_1^a \frac{\partial G_1^b}{\partial \overline{\zeta}}\right\rangle\right) + \mathcal{O}(\epsilon_d^2), \tag{21b}$$

where $\langle\,\rangle$ denotes averaging with respect to $\overline{\zeta}$. The unperturbed bounce guiding-center Poisson bracket is defined in terms of two arbitrary functions $\mathcal{F}$ and $\mathcal{G}$ on bounce guiding-center phase space $(\overline{\mathbf{y}},\overline{\mathbf{u}})$ as

$$\{\mathcal{F},\mathcal{G}\} = \frac{c\epsilon_d}{q}\frac{\partial \mathcal{F}}{\partial \overline{y}^a}\eta^{ab}\frac{\partial \mathcal{G}}{\partial \overline{y}^b} + \frac{\partial \mathcal{F}}{\partial \overline{u}^\alpha}\eta^{\alpha\beta}\frac{\partial \mathcal{G}}{\partial \overline{u}^\beta}, \tag{21c}$$

where $\eta^{\alpha\beta} \equiv \eta_{\alpha\beta}^{-1} = -\eta_{\alpha\beta}$.

From the unperturbed bounce guiding-center Hamiltonian (21b) and the unperturbed bounce guiding-center Poisson bracket (21c), the unperturbed bounce guiding-center Hamilton equations can now be written as

$$\frac{d\overline{y}^a}{dt} \equiv \{\overline{y}^a,\overline{H}_0\} = \epsilon_d \eta^{ab}\frac{c}{q}\frac{\partial \overline{H}_0}{\partial \overline{y}^b}, \tag{22a}$$

$$\frac{d\overline{J}}{dt} \equiv \{\overline{J},\overline{H}_0\} = -\frac{\partial \overline{H}_0}{\partial \overline{\zeta}} \equiv 0, \tag{22b}$$

$$\frac{d\overline{\zeta}}{dt} \equiv \{\overline{\zeta},\overline{H}_0\} = \frac{\partial \overline{H}_0}{\partial \overline{J}}. \tag{22c}$$

We note that since the unperturbed bounce guiding-center Hamiltonian $\overline{H}_0$ is asymptotically independent of the bounce guiding-center angle $\overline{\zeta}$, the longitudinal bounce guiding-center action $\overline{J}$ is an adiabatic invariant.

Last, we note that the bounce-guiding-center position $\overline{y}^a$ is simply the (bounce-motion) time-averaged position of the guiding-center position y^a, i.e., $\overline{y}^a \equiv \langle y^a\rangle$, where $\langle\,\rangle$ denotes bounce-angle averaging with respect to $\overline{\zeta}$, and thus

$$\Lambda_b^a \equiv y^a - \langle y^a\rangle = -\epsilon_d G_1^a(\overline{\mathbf{y}},\overline{\mathbf{u}}) + \mathcal{O}(\epsilon_d^2) \tag{23}$$

represents the bounce-angle dependent *bounce* radius. This definition of the bounce radius is entirely analogous to the definition of the gyroradius as the difference between the particle position and its (gyromotion) time-averaged position, the guiding-center position.

IV. NONLINEAR BOUNCE-CENTER HAMILTON EQUATIONS

In the presence of electromagnetic field fluctuations, the background magnetic field becomes perturbed. Depending on the characteristic time scales of the fluctuating fields, this situation typically may lead to the destruction of the guiding-center adiabatic invariants μ and/or $\overline{J}$. Here, the electromagnetic field fluctuations are represented by: the perturbed scalar potential $\delta\phi$, the parallel component of the perturbed vector potential $\delta A_\parallel(\equiv \hat{\mathbf{b}}_0\cdot\delta\mathbf{A})$, and the parallel component of the perturbed magnetic field $\delta B_\parallel(\equiv \hat{\mathbf{b}}_0\cdot\nabla\times\delta\mathbf{A})$. We shall assume that the characteristic mode frequency ω is much smaller than the bounce frequency ω_b, i.e.,

$$\omega_b^{-1}\frac{\partial}{\partial t} \equiv \mathcal{O}(\epsilon_\omega), \tag{24}$$

where ϵ_ω is a small ordering parameter; we henceforth set ϵ_d equal to one for clarity.

The perturbed guiding-center phase-space Lagrangian can be written as

$$\overline{\gamma} = \overline{\gamma}_0 + \epsilon_\delta \overline{\gamma}_1 + \epsilon_\delta^2 \overline{\gamma}_2 + \cdots, \tag{25}$$

where the first-order guiding-center phase-space Lagrangian is[11]

$$\overline{\gamma}_1 \equiv \left(\frac{q}{c}\,\delta\overline{A}_\parallel \frac{\partial s}{\partial \overline{u}^\alpha}\right) d\overline{u}^\alpha - (q\,\delta\overline{\phi} + \overline{\mu}\,\delta\overline{B}_\parallel)\,dt. \tag{26}$$

Here, dependence on the fast bounce-angle $\overline{\zeta}$ is re-introduced in $\overline{\gamma}_n$ $(n \geqslant 1)$ because the perturbation fields $(\delta\overline{\phi}, \delta\overline{A}_\parallel, \delta\overline{B}_\parallel)$ depend on $\overline{\zeta}$ through $s(\overline{\mathbf{u}}) \equiv s(\mathbf{u})$ (to lowest order in ϵ_d) and $\mathbf{y} \equiv \overline{\mathbf{y}} + \mathbf{\Lambda}_b$. For example, the perturbed scalar potential $\delta\overline{\phi}(\overline{\mathbf{y}},\overline{\mathbf{u}})$ is defined as

$$\delta\overline{\phi}(\overline{\mathbf{y}},\overline{\mathbf{u}};t) \equiv \delta\phi(\overline{\mathbf{y}} + \mathbf{\Lambda}_b, s(\overline{\mathbf{u}});t), \tag{27}$$

and, using this expression, we therefore find

$$\frac{\partial \delta \overline{\phi}}{\partial \overline{y}^a} \equiv \frac{\partial \delta \phi}{\partial y^a},$$

$$\frac{\partial \delta \overline{\phi}}{\partial \overline{u}^\alpha} \equiv \frac{\partial s}{\partial \overline{u}^\alpha} \frac{\partial \delta \phi}{\partial s} + \mathcal{O}(\epsilon_d).$$

In what follows, we make no assumptions about the orderings of the parallel and perpendicular wave numbers. In (26), the gyrocenter magnetic moment

$$\overline{\mu} \equiv \mu + \epsilon_\delta \left[\frac{q}{B} \boldsymbol{\rho} \cdot \overline{\boldsymbol{\nabla}} \left(\delta\overline{\phi} - \frac{v_\parallel}{c} \delta \overline{A}_\parallel \right) - \mu \frac{\delta \overline{B}_\parallel}{B} \right] + \mathcal{O}(\epsilon_\delta^2) \tag{28a}$$

is an adiabatic invariant for the low-frequency nonlinear gyrocenter Hamiltonian dynamics[11] while μ is the (unperturbed) guiding-center magnetic moment and $\boldsymbol{\rho}$ is the gyroradius. The second-order perturbed guiding-center phase-space Lagrangian can be written as $\overline{\gamma}_2 \equiv -\overline{H}_2 dt$, where

$$\overline{H}_2 \equiv -\frac{mc^2}{2B_0^2} \left| \overline{\boldsymbol{\nabla}}_\perp \left(\delta\overline{\phi} - \frac{v_\parallel}{c} \delta \overline{A}_\parallel \right) \right|^2 - \delta \overline{\mathbf{A}}_\perp \cdot \frac{\mathbf{b}_0}{B_0} \times \overline{\boldsymbol{\nabla}}_\perp \left(\delta\overline{\phi} - \frac{v_\parallel}{c} \delta \overline{A}_\parallel \right) \tag{28b}$$

is the second-order gyrocenter Hamiltonian (in the limit $\rho^2 k_\perp^2 \ll 1$).

The new bounce-gyrocenter phase-space Lagrangian is chosen to be of the form

$$\hat{\gamma} \equiv \frac{q}{2c} \eta_{ab} \hat{y}^a d\hat{y}^b + \hat{J}\, d\hat{\zeta} - \hat{H}\, dt, \tag{29a}$$

i.e., all the electromagnetic perturbation effects have been transferred to the bounce-gyrocenter Hamiltonian

$$\hat{H}(\hat{\mathbf{y}},t;\hat{J}) \equiv \hat{H}_0 + \epsilon_\delta \hat{H}_1 + \epsilon_\delta^2 \hat{H}_2, \tag{29b}$$

where terms of order ϵ_δ^3, $\epsilon_d \epsilon_\delta$, and $\epsilon_\omega \epsilon_\delta$ are omitted. The perturbed bounce-gyrocenter (bgy) Hamilton equations become

$$\frac{d\hat{y}^a}{dt} \equiv \{\hat{y}^a, \hat{H}\} = \frac{c}{q} \eta^{ab} \frac{\partial \hat{H}}{\partial \hat{y}^b}, \tag{30a}$$

$$\frac{d\hat{J}}{dt} \equiv \{\hat{J}, \hat{H}\} = -\frac{\partial \hat{H}}{\partial \hat{\zeta}} \equiv 0, \tag{30b}$$

$$\frac{d\hat{\zeta}}{dt} \equiv \{\hat{\zeta}, \hat{H}\} = \frac{\partial \hat{H}}{\partial \hat{J}}, \tag{30c}$$

where the bgy Poisson bracket is given for two arbitrary functions $\mathcal{F}(\hat{\mathbf{y}},\hat{\mathbf{u}})$ and $\mathcal{G}(\hat{\mathbf{y}},\hat{\mathbf{u}})$ as

$$\{\mathcal{F},\mathcal{G}\} \equiv \frac{c}{q} \frac{\partial \mathcal{F}}{\partial \hat{y}^a} \eta^{ab} \frac{\partial \mathcal{G}}{\partial \hat{y}^b} + \frac{\partial \mathcal{F}}{\partial \hat{u}^\alpha} \eta^{\alpha\beta} \frac{\partial \mathcal{G}}{\partial \hat{u}^\beta}. \tag{31}$$

By having removed the bounce-angle dependence from the phase-space Lagrangian (29a), we have restored the adiabatic invariance of the bounce action, now given as $\hat{J} \equiv \overline{J} + \epsilon_\delta \overline{G}_1^J + \cdots$. To derive explicit expressions for $\hat{H}_n$ $(n \geq 1)$, we proceed by the phase-space Lagrangian Lie perturbation method.[7,11]

A. First-order analysis

The relation between the old first-order bounce-guiding-center phase-space Lagrangian $\overline{\gamma}_1$ and the new first-order bounce-gyrocenter phase-space Lagrangian $\hat{\gamma}_1$ is expressed as

$$\hat{\gamma}_1 = \overline{\gamma}_1 - (\overline{G}_1 \cdot d\hat{\gamma}_0 - d\hat{\gamma}_0 \cdot \overline{G}_1) + d\overline{S}_1 \equiv -\hat{H}_1\, dt, \tag{32a}$$

where $\overline{G}_1$ is the first-order generating vector field for the transformation $(\overline{\mathbf{y}}, \overline{\mathbf{u}}) \to (\hat{\mathbf{y}}, \hat{\mathbf{u}})$, $\overline{S}_1$ is an arbitrary phase-space gauge function, and the new first-order Hamiltonian is

$$\hat{H}_1 = \overline{H}_1 - \overline{G}_1 \cdot d\hat{H}_0 - \partial_t \overline{S}_1. \tag{32b}$$

The requirement that $\hat{\gamma}_1 \equiv -\hat{H}_1\, dt$ dictates that the components of the first-order generating vector field be

$$\overline{G}_1^a \equiv \{\overline{S}_1, \hat{y}^a\}, \tag{33a}$$

$$\overline{G}_1^\alpha \equiv \{\overline{S}_1, \hat{u}^\alpha\} + \frac{q}{c} \delta \overline{A}_\parallel \{s, \hat{u}^\alpha\}. \tag{33b}$$

When these components are substituted into (32b), with $\overline{H}_1 = q\,\delta\overline{\phi} + \overline{\mu}\,\delta\overline{B}_\parallel$ and $\overline{G}_1 \cdot d\hat{H}_0 \equiv \{\overline{S}_1, \hat{H}_0\} + (q/c)\,\delta\overline{A}_\parallel v_\parallel$, where $v_\parallel \equiv \{s, \hat{H}_0\}$, we obtain

$$\hat{H}_1 = q\,\delta\overline{\phi} + \overline{\mu}\,\delta\overline{B}_\parallel - \left(\frac{\hat{d}\overline{S}_1}{dt} + \frac{q}{c} \delta\overline{A}_\parallel v_\parallel \right) \equiv \overline{K}_1 - \frac{\hat{d}\overline{S}_1}{dt}, \tag{34}$$

where $\hat{d}\overline{S}_1/dt \equiv \partial_t \overline{S}_1 + \{\overline{S}_1, \hat{H}_0\}$. The right-hand side of this equation has both bounce-angle independent and dependent parts. By design, we want the new first-order Hamiltonian $\hat{H}_1$ to be bounce-angle independent and hence we set

$$\hat{H}_1 \equiv \langle \overline{K}_1 \rangle = \left\langle q\,\delta\overline{\phi} + \overline{\mu}\,\delta\overline{B}_\parallel - \frac{q}{c} \delta\overline{A}_\parallel v_\parallel \right\rangle, \tag{35}$$

where the bounce-angle averaging $\langle\,\rangle$ is now with respect to $\hat{\zeta}$.

Because the Hamiltonian dynamics is independent of the phase-space gauge functions $\overline{S}_n$ $(n \geq 1)$, we choose $\langle \overline{S}_n \rangle \equiv 0$. To lowest order in the bounce-kinetic ordering (24), we find

$$\frac{\hat{d}\overline{S}_1}{dt} = \omega_b \frac{\partial \overline{S}_1}{\partial \hat{\zeta}} = \overline{K}_1 - \langle \overline{K}_1 \rangle \equiv \tilde{\overline{K}}_1, \tag{36}$$

whose solution is $\overline{S}_1 \equiv \omega_b^{-1} \int \tilde{\overline{K}}_1\, d\hat{\zeta}$. To first order in ϵ_δ, the phase-space transformation from $(\overline{\mathbf{y}}, \overline{\mathbf{u}})$ to $(\hat{\mathbf{y}}, \hat{\mathbf{u}})$ is represented by

$$\begin{aligned} \overline{y}^a &= \hat{y}^a - \epsilon_\delta \{\overline{S}_1, \hat{y}^a\} + \cdots, \\ \overline{u}^\alpha &= \hat{u}^\alpha - \epsilon_\delta \left(\{\overline{S}_1, \hat{u}^\alpha\} + \frac{q}{c} \delta\overline{A}_\parallel \{s, \hat{u}^\alpha\} \right) + \cdots. \end{aligned} \tag{37}$$

The main advantage of the phase-space Lagrangian Lie perturbation method is its algorithmic approach: one can continue the perturbation to arbitrary orders in ϵ_δ. To derive nonlinear perturbed Hamiltonian dynamics, we now proceed to derive terms of order ϵ_δ^2 in the phase-space Lagrangian.

B. Second-order analysis

The relation between the old second-order bounce-guiding-center phase-space Lagrangian $\overline{\gamma_2}$ and the new second-order bounce-gyrocenter phase-space Lagrangian $\hat{\gamma}_2$ is[11]

$$\hat{\gamma}_2 = \overline{\gamma_2} - (\overline{G_2}\cdot d\hat{\gamma}_0 - d\hat{\gamma}_0\cdot\overline{G_2}) - \frac{1}{2}[\overline{G_1}\cdot d(\overline{\gamma_1}+\hat{\gamma}_1) - d(\overline{\gamma_1}+\hat{\gamma}_1)\cdot\overline{G_1}] + d\overline{S_2}, \tag{38a}$$

where $\overline{G_2}$ is the second-order generating vector field for the transformation $(\overline{\mathbf{y}},\overline{\mathbf{u}})\rightarrow(\hat{\mathbf{y}},\hat{\mathbf{u}})$, $\overline{S_2}$ is an arbitrary phase-space gauge function, and the new second-order Hamiltonian is

$$\hat{H}_2 = \overline{H_2} - \overline{G_2}\cdot d\hat{H}_0 - \frac{1}{2}\overline{G_1}\cdot d(\overline{H_1}+\hat{H}_1) - \partial_t\overline{S_2}. \tag{38b}$$

The requirement that $\hat{\gamma}_2 \equiv -\hat{H}_2\,dt$ dictates that the components of the second-order generating vector field be

$$\overline{G_2^a} \equiv \{\overline{S_2},\hat{y}^a\} + \frac{q}{2c}\overline{G_1^\alpha}\left\{\delta\overline{A_\parallel}\frac{\partial s}{\partial\hat{u}^\alpha},\hat{y}^a\right\}, \tag{39a}$$

$$\overline{G_2^\alpha} \equiv \{\overline{S_2},\hat{u}^\alpha\} + \frac{q}{2c}\left[\overline{G_1^\beta}\left\{\delta\overline{A_\parallel}\frac{\partial s}{\partial\hat{u}^\beta},\hat{u}^\alpha\right\} - \overline{G_1}\cdot d(\delta\overline{A_\parallel}\{s,\hat{u}^\alpha\})\right]. \tag{39b}$$

When these components are substituted into (38b), we obtain

$$\overline{G_2}\cdot d\hat{H}_0 = \{\overline{S_2},\hat{H}_0\} + \frac{q}{2c}\overline{G_1^\sigma}[\{(\delta\overline{A_\parallel}\partial_\sigma s),\hat{H}_0\} + \eta^{\alpha\beta}\partial_\alpha\hat{H}_0\partial_\sigma(\delta\overline{A_\parallel}\partial_\beta s)]. \tag{40}$$

After some manipulations, (40) becomes

$$\overline{G_2}\cdot d\hat{H}_0 = \left\{\left(\overline{S_2} - \frac{q}{2c}\delta\overline{A_\parallel}\{s,\overline{S_1}\}\right),\hat{H}_0\right\} + \frac{q}{2c}\delta\overline{A_\parallel}\left(\{s,\{\overline{S_1},\hat{H}_0\}\} + \frac{q}{c}\delta\overline{A_\parallel}\{s,\{s,\hat{H}_0\}\}\right) - \frac{q}{2c}\left(\{\overline{S_1},\delta\overline{A_\parallel}\{s,\hat{H}_0\}\} + \frac{q}{c}\delta\overline{A_\parallel}\{s,\delta\overline{A_\parallel}\{s,\hat{H}_0\}\}\right),$$

so that expression (38b) can be written as

$$\hat{H}_2 = \overline{H_2} - \left\{\left(\overline{S_2} - \frac{q}{2c}\delta\overline{A_\parallel}\{s,\overline{S_1}\}\right),\hat{H}_0\right\} - \frac{q^2}{2mc^2}(\delta\overline{A_\parallel})^2 - \frac{1}{2}\left\{\overline{S_1},\left(\overline{H_1}+\hat{H}_1 - \frac{q}{c}\delta\overline{A_\parallel}\{s,\hat{H}_0\}\right)\right\} - \frac{q}{2c}\delta\overline{A_\parallel}\left\{s,\left(\overline{H_1}+\hat{H}_1+\{\overline{S_1},\hat{H}_0\} - \frac{q}{c}\delta\overline{A_\parallel}\{s,\hat{H}_0\}\right)\right\}, \tag{41}$$

where we used the canonical relation (17), or $\{s,\{s,\hat{H}_0\}\}\equiv 1/m$, in the third term.

The right-hand side of (41) has both bounce-angle independent and dependent parts. By design, we want the new second-order Hamiltonian to be bounce-angle independent and hence we set

$$\hat{H}_2 \equiv \langle\overline{H_2}\rangle + \frac{q^2}{2mc^2}\langle(\delta\overline{A_\parallel})^2\rangle - \frac{1}{2}\langle\{\overline{S_1},\overline{K_1}\}\rangle, \tag{42}$$

where $\overline{K_1}$ is defined in (34), $\langle\{\cdots,\hat{H}_0\}\rangle = 0$ to lowest order in ϵ_ω and ϵ_d, and we used the identity

$$\overline{H_1} + \hat{H}_1 + \{\overline{S_1},\hat{H}_0\} - \frac{q}{c}\delta\overline{A_\parallel}\{s,\hat{H}_0\} = 2\overline{K_1}. \tag{43}$$

The second-order phase-space gauge function $\overline{S_2}$ in (41) is not needed in what follows and will not be given here.

C. Nonlinear bounce-gyrocenter Hamiltonian dynamics

The nonlinear bounce-gyrocenter Hamiltonian is expressed as

$$\hat{H} \equiv \hat{H}_0 + \epsilon_\delta\left\langle q\,\delta\overline{\phi} + \overline{\mu}\,\delta\overline{B_\parallel} - \frac{qv_\parallel}{c}\delta\overline{A_\parallel}\right\rangle + \epsilon_\delta^2\left[\langle\overline{H_2}\rangle + \frac{q^2}{2mc^2}\langle(\delta\overline{A_\parallel})^2\rangle - \frac{1}{2}\langle\{\overline{S_1},\overline{K_1}\}\rangle\right]. \tag{44}$$

This expression generalizes the previous works of Gang and Diamond[12] and Fong and Hahm,[13] who considered electrostatic perturbations only. The nonlinear bounce-gyrocenter Hamilton equations presented here contain terms associated with full electromagnetic perturbations and include classical ($\langle\overline{H_2}\rangle$) and neoclassical ($\langle\{\overline{S_1},\overline{K_1}\}\rangle$) terms.

V. SUMMARY

We now summarize our work and discuss possible extensions and applications. We have derived nonlinear reduced Hamilton equations describing the bounce-gyrocenter dynamics of trapped particles in nonuniform magnetized plasmas in the presence of low-frequency electromagnetic field fluctuations. The derivation was done within the context of a general magnetic field geometry defined in terms of magnetic coordinates; these equations can therefore be used in a variety of applications in fusion and space plasma physics, such as the dynamics of fusion alphas or beam ions in high-temperature tokamak plasmas or the dynamics of trapped protons in planetary magnetoplasmas.

The characteristic mode frequency ω for these fluctuations was assumed to be much smaller than the gyrofrequency (ω_c) and bounce-frequency (ω_b). This time-scale ordering allowed the asymptotic elimination of the fast gyromotion and bounce-motion degrees of freedom from the exact Hamiltonian dynamics; the asymptotic elimination of fast degrees of freedom is performed by using the phase-space Lagrangian Lie-perturbation method.[7] For low-frequency electromagnetic field fluctuations in general magnetic geometry, the asymptotic elimination of the fast gyromotion time scale was carried out previously by Brizard.[11] The asymptotic elimination of the fast bounce-motion time scale, on the other hand, was described in the present paper. Here, the phase-space transformations (19) and (37) were explicitly written in terms of first-order generating vector fields (G_1^a, G_1^α) and $(\overline{G}_1^a, \overline{G}_1^\alpha)$, respectively.

The equations presented here do not take into account the self-consistent response of the electromagnetic field to the presence of a trapped-particle population. Future work will consider the derivation of the low-frequency bounce-gyrocenter Vlasov–Maxwell equations in which the charge density and current in the Maxwell equations are expressed as moments of the bounce-gyrocenter Vlasov distribution function. For this purpose, we plan to derive these equations from a variational principle, from which explicit conservation laws will also be derived (through the Noether method). The variational principle will make explicit use of the phase-space transformations described here and in Ref. 11.

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APPENDIX: EXAMPLES OF MAGNETIC GEOMETRIES

In this appendix we consider two explicit magnetic geometries: the axisymmetric tokamak geometry (for applications in fusion plasma physics) and the asymmetric geomagnetic geometry (for applications in space plasma physics).

1. Axisymmetric tokamak geometry

As a first example, we consider a general axisymmetric tokamak magnetic field.[2] This magnetic field has toroidal $(B^\varphi \equiv \mathbf{B}\cdot\nabla\varphi)$ and poloidal $(B^\theta \equiv \mathbf{B}\cdot\nabla\theta)$ components, where φ and θ are the toroidal and poloidal angles, respectively, and the magnetic field possesses toroidal symmetry, i.e., its components B^φ and B^θ are independent of the toroidal angle φ.

The Euler-potential representation

$$\mathbf{B} \equiv \nabla\xi \times \nabla\psi \tag{A1}$$

for an axisymmetric tokamak magnetic field is given in terms of the poloidal flux function ψ and the angle-like Euler potential

$$\xi(\psi,\theta,\varphi) \equiv \varphi - \nu(\psi,\theta), \tag{A2}$$

where θ is the poloidal angle and

$$\frac{\partial\nu(\psi,\theta)}{\partial\theta} \equiv Q(\psi,\theta) \tag{A3}$$

represents the local safety factor (i.e., $Q \equiv \mathbf{B}\cdot\nabla\varphi/\mathbf{B}\cdot\nabla\theta$ represents the amount of twist in the magnetic field lines). The covariant representation (4) for the axisymmetric tokamak magnetic field, on the other hand, is written as

$$\mathbf{B} \equiv I(\psi,\theta)\nabla\varphi + g(\psi,\theta)\nabla_\psi\theta, \tag{A4}$$

where $\nabla_\psi\theta \equiv \nabla\theta - \nabla\psi(\nabla\theta\cdot\nabla\psi/|\nabla\psi|^2)$ while $\nabla\psi\cdot\nabla\varphi \equiv 0$ (by axisymmetry). In this representation, the local safety factor is $Q = I|\nabla\varphi|^2/(g|\nabla_\psi\theta|^2)$. Last, the tokamak magnetic field can also be expressed as

$$\mathbf{B} \equiv \nabla\Psi \times \nabla\theta + \nabla\varphi \times \nabla\psi, \tag{A5}$$

where $\Psi(\psi,\theta)$ is the toroidal flux function and $Q \equiv \partial\Psi/\partial\psi$. We note that in the covariant representation $\mathbf{A} \equiv \Psi\nabla\theta - \psi\nabla\varphi$, the magnetic helicity is $\mathbf{A}\cdot\mathbf{B} = \Psi\mathcal{J}^{-1} - \mathbf{B}\cdot\nabla(\varphi\psi)$, where $\mathcal{J}^{-1} \equiv \nabla\psi\times\nabla\theta\cdot\nabla\varphi$. If the magnetic field has the Euler representation (A1), then $\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\nabla\alpha$, where $\alpha \equiv \beta - \varphi\psi$ and $\Psi \equiv \partial_\theta\beta$ and $\nu \equiv \partial_\psi\beta$ (since $\partial\Psi/\partial\psi \equiv Q \equiv \partial\nu/\partial\theta$).

We now construct the vectors $\nabla\Psi^i$ and $\partial\mathbf{X}/\partial\Psi^i$ for axisymmetric tokamak geometry. From (A2), we find

$$\nabla\xi \equiv \nabla\varphi - Q(\psi,\theta)\nabla_\psi\theta - D(\psi,\theta)\nabla\psi, \tag{A6}$$

where the function

$$D(\psi,\theta) \equiv \frac{\nabla\nu\cdot\nabla\psi}{|\nabla\psi|^2} = \frac{\partial\nu}{\partial\psi} + Q\,\frac{\nabla\theta\cdot\nabla\psi}{|\nabla\psi|^2} \tag{A7}$$

includes magnetic shear $(\partial\nu/\partial\psi \neq 0)$ and includes the Shafranov shift in the magnetic surfaces $(\nabla\theta\cdot\nabla\psi \neq 0)$. The parallel spatial coordinate $s(\psi,\theta,\varphi)$ is defined by the covariant expression

$$\frac{\partial\mathbf{X}}{\partial s} \equiv \hat{\mathbf{b}} = B^{-1}(I\nabla\varphi + g\nabla_\psi\theta). \tag{A8}$$

The complement ∇s to this equation is

$$\nabla s \equiv \hat{\mathbf{b}} + \frac{gD}{QB}\nabla\psi, \tag{A9}$$

so that in axisymmetric tokamak geometry

$$\mathcal{R}_\xi = \hat{\mathbf{b}}\cdot\frac{\partial\mathbf{X}}{\partial\xi} \equiv 0, \quad \mathcal{R}_\psi = \hat{\mathbf{b}}\cdot\frac{\partial\mathbf{X}}{\partial\psi} \equiv -\frac{gD}{QB}. \tag{A10}$$

Thus the vectors $\nabla\psi$, $\nabla\xi$ [(A6)], $\partial\mathbf{X}/\partial s$ [(A8)], and $\partial\mathbf{X}/\partial y^a$ [(A9)] can now be combined with the contravariant vectors

$$\frac{\partial\mathbf{X}}{\partial\psi} = -\left(\frac{gD}{QB}\right)\hat{\mathbf{b}} + \frac{\mathbf{B}}{B^2}\times\nabla\xi, \quad \frac{\partial\mathbf{X}}{\partial\xi} = -\frac{\mathbf{B}}{B^2}\times\nabla\psi, \tag{A11}$$

to define the axisymmetric tokamak geometry.

2. Asymmetric geomagnetic geometry

Our next example involves the geomagnetic field in the region below 10 R_E (R_E is the Earth's radius). As a result of the solar wind, the geomagnetic field exhibits a day–night asymmetry.[14,15] Yet, as was shown by Stern,[16] the geomagnetic field has the following Euler–Clebsch representations:

$$\mathbf{B} \equiv \nabla\xi \times \nabla\psi \equiv \nabla\chi, \tag{A12}$$

where the last expression represents the fact that $\nabla\times\mathbf{B} = 0$ for the geomagnetic field. To lowest order, the geomagnetic field is represented as a pure dipole field with Euler/Clebsch potentials

$$\begin{aligned} \xi_0 &\equiv \varphi, \\ \psi_0 &\equiv \mathcal{M}r^{-1}\sin^2\theta, \\ \chi_0 &\equiv \mathcal{M}r^{-2}\cos\theta, \end{aligned} \tag{A13}$$

where $\mathcal{M} \equiv B_0 R_\mathrm{E}^3$ denotes the dipole moment and (r,θ,φ) are spherical coordinates in the tilted-dipole frame.[17]

The asymmetry in the geomagnetic field enters at higher multipole order. In the two-parameter first-order model presented by Stern,[16] we find

$$\psi = \mathcal{M} r^{-1} \sin^2\theta \left[(1-\epsilon\rho^3) + 2\delta\rho^4 \left(1 - \frac{7}{3}\sin^2\theta \right) \cos\varphi / \sin\theta \right],$$

$$\xi = \varphi - \delta\rho^4 \sin\varphi / \sin\theta, \tag{A14}$$

$$\chi = \mathcal{M} r^{-2} \cos\theta [(1+2\epsilon\rho^3) + 7\delta\rho^4 \cos\varphi \sin\theta],$$

where $\rho \equiv r/r_E$ is the normalized (dimensionless) radius and ϵ and δ are two small dimensionless parameters. This model is a good approximation of the geomagnetic field up to 10 R_E. The compressed-dipole model is obtained from (A14) by setting $\delta=0$. When $\delta \neq 0$, the night side ($\pi/2 < \varphi < 3\pi/2$) opens up and allows field lines to extend to infinity; these potentials yield the so-called distorted dipole model.[18] Finally, introducing the magnetic coordinates (ψ, ξ, s), where from (8b), we find

$$B \equiv \frac{\partial\chi}{\partial s}, \quad \mathcal{R}_a \equiv \frac{1}{B} \frac{\partial\chi}{\partial y^a}. \tag{A15}$$

These expressions can now be used to define the asymmetric geomagnetic geometry.

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